

Parabolic log convergent isocrystals

Atsushi Shiho *

Abstract

In this paper, we introduce the notion of parabolic log convergent isocrystals on smooth varieties endowed with a simple normal crossing divisor, which is a kind of p -adic analogue of the notion of parabolic bundles on smooth varieties defined by Seshadri, Maruyama-Yokogawa, Iyer-Simpson, Borne. We prove that the equivalence between the category of p -adic representations of the fundamental group and the category of unit-root convergent F -isocrystals (proven by Crew) induces the equivalence between the category of p -adic representations of the tame fundamental group and the category of semisimply adjusted parabolic unit-root log convergent F -isocrystals. We also prove equivalences which relate categories of log convergent isocrystals on certain fine log algebraic stacks with some conditions and categories of adjusted parabolic log convergent isocrystals with some conditions. We also give an interpretation of unit-rootness in terms of the generic semistability with slope 0. Our result can be regarded as a p -adic analogue of some results of Seshadri, Mehta-Seshadri, Iyer-Simpson and Borne.

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*Graduate School of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, JAPAN. E-mail address: shiho@ms.u-tokyo.ac.jp Mathematics Subject Classification (2000): 12H25, 14F35.

Introduction

For a proper smooth curve X over \mathbb{C} of genus ≥ 2 , Narasimhan-Seshadri [32] proved the equivalence

$$(0.1) \quad \left(\begin{array}{c} \text{irreducible unitary} \\ \text{representation of } \pi_1(X^{\text{an}}) \end{array} \right) \xrightarrow{=} \left(\begin{array}{c} \text{vector bundle on } X \\ \text{stable of degree 0} \end{array} \right).$$

(Here, for a scheme S of finite type over \mathbb{C} , S^{an} denotes the analytic space associated to S .) In the case where a given smooth curve X over \mathbb{C} is open with compactification \overline{X} , there are two approaches to prove an analogue of the above result: In [37], Seshadri took a ‘stacky’ approach and proved the equivalence

$$(0.2) \quad \left(\begin{array}{c} \text{irreducible unitary} \\ \text{representation of } \pi_1(X^{\text{an}}) \end{array} \right) \xrightarrow{=} \lim_{Y \rightarrow X \in \mathcal{G}_X} \left(\begin{array}{c} \text{vector bundle on } [\overline{Y}/G_Y] \\ \text{stable of degree 0} \end{array} \right),$$

where \mathcal{G}_X denotes the category of finite étale Galois covering of X , \overline{Y} denotes the smooth compactification of Y , $G_Y := \text{Aut}(Y/X)$ and $[\overline{Y}/G_Y]$ denotes the quotient stack. (Note that a vector bundle on $[\overline{Y}/G_Y]$ is nothing but a G_Y -equivariant vector bundle on \overline{Y} .) On the other hand, in [27], Mehta-Seshadri took a ‘parabolic’ approach: They defined the notion of parabolic vector bundles on (\overline{X}, Z) (where $Z := \overline{X} \setminus X$) and proved the following equivalence:

$$(0.3) \quad \left(\begin{array}{c} \text{irreducible unitary} \\ \text{representation of } \pi_1(X^{\text{an}}) \end{array} \right) \xrightarrow{=} \left(\begin{array}{c} \text{parabolic vector} \\ \text{bundle on } (\overline{X}, Z) \\ \text{parabolic stable} \\ \text{of parabolic degree 0} \end{array} \right).$$

Note that the notion of parabolic vector bundles and the moduli of them are studied also by Maruyama-Yokogawa [25], including the higher-dimensional case.

Let us recall that there exists another ‘stacky’ interpretation: For an open immersion $X \hookrightarrow \overline{X}$ of smooth varieties over \mathbb{C} such that $Z := \overline{X} \setminus X$ is a simple normal crossing divisor, Iyer-Simpson [14] and Borne [3] [4] introduced the notion of ‘stack of roots’ $(\overline{X}, Z)^{1/n}$ ($n \in \mathbb{N}$) and established the equivalence

$$(0.4) \quad \left(\begin{array}{c} \text{parabolic vector} \\ \text{bundle on } (\overline{X}, Z) \end{array} \right) \xrightarrow{=} \lim_n \left(\begin{array}{c} \text{vector bundle} \\ \text{on } (\overline{X}, Z)^{1/n} \end{array} \right)$$

such that the parabolic degree on the left hand side coincides with the degree on the right hand side. (There is also a related work by Biswas [2].) So, when X is a curve, the equivalences (0.3) and (0.4) imply the equivalence

$$(0.5) \quad \left(\begin{array}{c} \text{irreducible unitary} \\ \text{representation of } \pi_1(X^{\text{an}}) \end{array} \right) \xrightarrow{=} \varinjlim_n \left(\begin{array}{c} \text{vector bundle on } (\overline{X}, Z)^{1/n} \\ \text{stable of degree 0} \end{array} \right).$$

A generalization of (0.1) to higher-dimensional case and non-unitary case (where Higgs bundle appears) is established by many people including Donaldson [11], Mehta-Ramanathan [26], Uhlenbeck-Yau [50], Corlette [7] and Simpson [39]. As for open case, a generalization of (0.3) to higher-dimensional non-unitary case (where parabolic Higgs bundle appears) is given also by many people including Simpson [40], Jost-Zuo [15] and Mochizuki [28] [29] [30].

Now let us turn to the p -adic situation. Let p be a prime, let q be a fixed power of p and let K be a complete discrete valuation field of characteristic zero with ring of integers O_K and perfect residue field k of characteristic $p > 0$ containing \mathbb{F}_q . Assume moreover that we have an endomorphism $\sigma : K \rightarrow K$ which respects O_K and which lifts the q -th power map on k . Let K^σ be the fixed field of σ . Then, for a connected smooth k -variety X (not necessarily proper), Crew [9] proved the equivalence

$$(0.6) \quad G : \text{Rep}_{K^\sigma}(\pi_1(X)) \xrightarrow{=} F\text{-Isoc}(X)^\circ$$

between the category $\text{Rep}_{K^\sigma}(\pi_1(X))$ of finite dimensional continuous representation of the algebraic fundamental group $\pi_1(X)$ of X over K^σ and the category $F\text{-Isoc}(X)^\circ$ of unit-root convergent F -isocrystals on X over K . When X is proper smooth, we regard (0.6) as an analogue of (0.1). (When X is not proper, the categories in (0.6) are considered to be too big.)

Now let $X \hookrightarrow \overline{X}$ be an open immersion of smooth k -varieties such that $Z := \overline{X} \setminus X$ is a simple normal crossing divisor. Weng [51] raised a question on the construction of the p -adic analogues of (0.2) and (0.3) for (X, \overline{X}) (at least for curves), starting from the equivalence (0.6). In this paper, we will prove several equivalences which can be regarded as p -adic analogues of (0.2), (0.3) and (0.5) (so we think it answers the question of Weng in some sense).

Let us explain our main results more precisely. Let X, \overline{X} be as above and let $\overline{X} \setminus X =: Z = \bigcup_{i=1}^r Z_i$ be the decomposition of Z into irreducible components. For $1 \leq i \leq r$, let v_i be the discrete valuation of $k(X)$ corresponding to the generic point of Z_i , let $k(X)_{v_i}$ be the completion of $k(X)$ with respect to v_i and let I_{v_i} be the inertia group of $k(X)_{v_i}$. (Then we have homomorphisms $I_{v_i} \rightarrow \pi_1(X)$ which are well-defined up to conjugate.) Let us define $\text{Rep}_{K^\sigma}^{\text{fin}}(\pi_1(X))$ by

$$\text{Rep}_{K^\sigma}^{\text{fin}}(\pi_1(X)) := \{\rho \in \text{Rep}_{K^\sigma}(\pi_1(X)) \mid \forall i, \rho|_{I_{v_i}} \text{ has finite image}\}.$$

Then first we prove the equivalence

$$(0.7) \quad \text{Rep}_{K^\sigma}^{\text{fin}}(\pi_1(X)) \xrightarrow{=} \varinjlim_{Y \rightarrow X \in \mathcal{G}_X} F\text{-Isoc}([\overline{Y}^{\text{sm}}/G_Y])^\circ,$$

where \mathcal{G}_X is the category of finite etale Galois covering of X , \overline{Y} is the normalization of \overline{X} in $k(Y)$, \overline{Y}^{sm} is the smooth locus of \overline{Y} , $G_Y := \text{Aut}(Y/X)$, $[\overline{Y}^{\text{sm}}/G_Y]$ is the quotient stack and the right hand side is the limit of the category of unit-root convergent F -isocrystals on stacks $[\overline{Y}^{\text{sm}}/G_Y]$ (which we will define in Section 2). This is a p -adic analogue of (0.2). The above equivalence induces the equivalence

$$(0.8) \quad \text{Rep}_{K^\sigma}(\pi_1^t(X)) \xrightarrow{\cong} \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} F\text{-Isoc}([\overline{Y}^{\text{sm}}/G_Y])^\circ,$$

where $\pi_1^t(X)$ is the tame fundamental group of X (tamely ramified at the valuations v_i ($1 \leq i \leq r$)), \mathcal{G}_X^t is the category of finite etale Galois covering of X tamely ramified at v_i ($1 \leq i \leq r$) and the other notations are the same as before. Next, we prove the equivalence

$$(0.9) \quad \text{Rep}_{K^\sigma}(\pi_1^t(X)) \xrightarrow{\cong} \varinjlim_{(n,p)=1} F\text{-Isoc}((\overline{X}, Z)^{1/n})^\circ,$$

where the right hand side is the limit of the category of unit-root convergent F -isocrystals on stacks of roots $(\overline{X}, Z)^{1/n}$. This is a p -adic analogue of (0.5). Also, we introduce the category $\text{Par-}F\text{-Isoc}^{\log}(\overline{X}, Z)_{\mathbf{0}\text{-ss}}^\circ$ of semisimply adjusted parabolic unit-root log convergent F -isocrystals on (\overline{X}, Z) and prove the equivalence

$$(0.10) \quad \text{Rep}_{K^\sigma}(\pi_1^t(X)) \xrightarrow{\cong} \text{Par-}F\text{-Isoc}^{\log}(\overline{X}, Z)_{\mathbf{0}\text{-ss}}^\circ,$$

which is a p -adic analogue of (0.3). The key ingredients of the proof are results of Tsuzuki in [49] and results of the author in [44] and [47]. We also discuss the relations among the variants (without Frobenius structure, with log structure and with exponent condition) of the categories on the right hand side of (0.8), (0.9) and (0.10).

In the case where \overline{X} is liftable to a smooth formal scheme $\overline{\mathcal{X}}_\circ$ over $\text{Spf } W(k)$ together with a suitable lift of Frobenius endomorphism $F_\circ : \overline{\mathcal{X}}_\circ \rightarrow \overline{\mathcal{X}}_\circ$, the equivalence (0.6) of Crew factors through an equivalence of Katz ([17], see also [9])

$$G : \text{Rep}_{O_K^\sigma}(\pi_1(\overline{X})) \xrightarrow{\cong} F\text{-Latt}(\overline{\mathcal{X}})^\circ$$

between the category $\text{Rep}_{O_K^\sigma}(\pi_1(\overline{X}))$ of continuous representations of $\pi_1(\overline{X})$ to free O_K^σ -modules of finite rank (here $O_K^\sigma := K^\sigma \cap O_K$) and the category $F\text{-Latt}(\overline{\mathcal{X}})^\circ$ of unit-root F -lattices on $\overline{\mathcal{X}} := \overline{\mathcal{X}}_\circ \otimes_{W(k)} O_K$. We also prove in the paper that, when (\overline{X}, Z) lifts to a smooth formal scheme $(\overline{\mathcal{X}}_\circ, \mathcal{Z}_\circ)$ over $\text{Spf } W(k)$ endowed with a relative simple normal crossing divisor together with a lift of Frobenius endomorphism $F_\circ : (\overline{\mathcal{X}}_\circ, \mathcal{Z}_\circ) \rightarrow (\overline{\mathcal{X}}_\circ, \mathcal{Z}_\circ)$ (endomorphism as log formal schemes), there exist equivalences of the form

$$(0.11) \quad \text{Rep}_{O_K^\sigma}(\pi_1^t(X)) \xrightarrow{\cong} \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} F\text{-Latt}([\overline{\mathcal{Y}}^{\text{sm}}/G_Y])^\circ,$$

$$(0.12) \quad \mathrm{Rep}_{O_K^\sigma}(\pi_1^t(X)) \xrightarrow{=} \varinjlim_{(n,p)=1} F\text{-Latt}((\overline{\mathcal{X}}, \mathcal{Z})^{1/n})^\circ$$

(where $(\overline{\mathcal{X}}, \mathcal{Z}) := (\overline{\mathcal{X}}_\circ, \mathcal{Z}_\circ) \otimes_{W(k)} O_K$, $\overline{\mathcal{Y}}^{\mathrm{sm}}$ is a certain lift of the smooth locus $\overline{Y}^{\mathrm{sm}}$ of the normalization \overline{Y} of \overline{X} in $k(Y)$), the left hand sides are the category of continuous representations of $\pi_1^t(X)$ to free O_K^σ -modules of finite rank and the right hand sides are the limits of the categories of unit-root F -lattices on the ind-stacks $[\overline{\mathcal{Y}}^{\mathrm{sm}}/G_Y]$, $(\overline{\mathcal{X}}, \mathcal{Z})^{1/n}$, respectively) which can be proven in the same way as (0.8) and (0.9). Moreover, we will introduce the category $\mathrm{Par}\text{-}F\text{-Latt}(\overline{\mathcal{X}}, \mathcal{Z})^\circ$ of locally abelian parabolic unit-root F -lattices on $(\overline{\mathcal{X}}, \mathcal{Z})$ and prove the equivalence

$$(0.13) \quad \mathrm{Rep}_{O_K^\sigma}(\pi_1^t(X)) \xrightarrow{=} \mathrm{Par}\text{-}F\text{-Latt}(\overline{\mathcal{X}}, \mathcal{Z})^\circ.$$

Note that, in the p -adic equivalences we have explained above, the notion of ‘the stability of degree 0’ does not appear, which appears in (0.2), (0.3), (0.5). To see the p -adic analogue of this notion more clearly, we introduce the notion of generic semistability (gss) and the invariant μ for objects in the category $F\text{-Isoc}([\overline{\mathcal{Y}}^{\mathrm{sm}}/G_Y])$ (resp. $F\text{-Isoc}((\overline{\mathcal{X}}, \mathcal{Z})^{1/n})$, $\mathrm{Par}\text{-}F\text{-Isoc}^{\mathrm{log}}(\overline{\mathcal{X}}, \mathcal{Z})_\mathbf{0}$) of convergent F -isocrystals on $[\overline{\mathcal{Y}}^{\mathrm{sm}}/G_Y]$ (resp. convergent F -isocrystals on $(\overline{\mathcal{X}}, \mathcal{Z})^{1/n}$, adjusted parabolic log convergent F -isocrystals on $(\overline{\mathcal{X}}, \mathcal{Z})$) (where the notations are as in (0.7), (0.9) and (0.10)) and rewrite the equivalences (0.7), (0.8), (0.9) and (0.10) as

$$(0.14) \quad \mathrm{Rep}_{K^\sigma}^{\mathrm{fin}}(\pi_1(X)) \xrightarrow{=} \varinjlim_{Y \rightarrow X \in \mathcal{G}_X} F\text{-Isoc}([\overline{\mathcal{Y}}^{\mathrm{sm}}/G_Y])^{\mathrm{gss}, \mu=0},$$

$$(0.15) \quad \mathrm{Rep}_{K^\sigma}(\pi_1^t(X)) \xrightarrow{=} \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} F\text{-Isoc}([\overline{\mathcal{Y}}^{\mathrm{sm}}/G_Y])^{\mathrm{gss}, \mu=0},$$

$$(0.16) \quad \mathrm{Rep}_{K^\sigma}(\pi_1^t(X)) \xrightarrow{=} \varinjlim_{(n,p)=1} F\text{-Isoc}((\overline{\mathcal{X}}, \mathcal{Z})^{1/n})^{\mathrm{gss}, \mu=0},$$

$$(0.17) \quad \mathrm{Rep}_{K^\sigma}(\pi_1^t(X)) \xrightarrow{=} \mathrm{Par}\text{-}F\text{-Isoc}^{\mathrm{log}}(\overline{\mathcal{X}}, \mathcal{Z})_\mathbf{0}^{\mathrm{gss}, \mu=0},$$

where $^{\mathrm{gss}, \mu=0}$ means the subcategory consisting of generically semistable objects with $\mu = 0$. The proof is an easy application of some results of Katz [18] and Crew [8], [9]. Also, we introduce the notion of generic semistability (gss) and the invariant μ for objects in the \mathbb{Q} -linearization $F\text{-Latt}([\overline{\mathcal{Y}}^{\mathrm{sm}}/G_Y])_\mathbb{Q}$ (resp. $F\text{-Latt}((\overline{\mathcal{X}}, \mathcal{Z})^{1/n})_\mathbb{Q}$, $\mathrm{Par}\text{-}F\text{-Latt}(\overline{\mathcal{X}}, \mathcal{Z})$) of the category of F -lattices on $[\overline{\mathcal{Y}}^{\mathrm{sm}}/G_Y]$ (resp. F -lattices on $(\overline{\mathcal{X}}, \mathcal{Z})^{1/n}$, locally abelian parabolic F -lattices on $(\overline{\mathcal{X}}, \mathcal{Z})$) (where the notations are as in (0.11), (0.12) and (0.13)) and rewrite the \mathbb{Q} -linearization of the equivalences (0.11), (0.12) and (0.13) as

$$(0.18) \quad \mathrm{Rep}_{K^\sigma}(\pi_1^t(X)) \xrightarrow{=} \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} F\text{-Latt}([\overline{\mathcal{Y}}^{\mathrm{sm}}/G_Y])_\mathbb{Q}^{\mathrm{gss}, \mu=0},$$

$$(0.19) \quad \mathrm{Rep}_{K^\sigma}(\pi_1^t(X)) \xrightarrow{=} \varinjlim_{(n,p)=1} F\text{-Latt}((\overline{\mathcal{X}}, \mathcal{Z})^{1/n})_\mathbb{Q}^{\mathrm{gss}, \mu=0},$$

$$(0.20) \quad \mathrm{Rep}_{K^\sigma}(\pi_1^t(X)) \xrightarrow{=} \mathrm{Par}\text{-}F\text{-Latt}(\overline{\mathcal{X}}, \mathcal{Z})_\mathbb{Q}^{\mathrm{gss}, \mu=0}.$$

The equivalences (0.14)–(0.20) might be better p -adic analogues of (0.2), (0.3), (0.5), but they are not good in the following point: In the equivalences (0.14), (0.15), (0.16) and (0.17), an object in the category on the right hand side contains an isocrystal structure (= p -adic version of connection structure) unlike the equivalences (0.2), (0.3), (0.5). In the equivalences (0.18), (0.19) and (0.20), an object in the category on the right hand side contains a lattice structure (= p -adic version of metric) unlike the equivalences (0.2), (0.3), (0.5). To overcome this, we introduce the category $F\text{-Vect}([\overline{\mathcal{Y}}^{\text{sm}}/G_Y]_K)$ (resp. $F\text{-Vect}((\overline{\mathcal{X}}, \mathcal{Z})_K^{1/n})$, $\text{Par-}F\text{-Vect}((\overline{\mathcal{X}}, \mathcal{Z})_K)$) of ‘ F -vector bundles on rigid analytic stack $[\overline{\mathcal{Y}}^{\text{sm}}/G_Y]_K$ ’ (resp. ‘ F -vector bundles on rigid analytic stack $(\overline{\mathcal{X}}, \mathcal{Z})_K^{1/n}$ ’, ‘locally abelian parabolic F -vector bundles on log rigid analytic space $(\overline{\mathcal{X}}, \mathcal{Z})_K$ ’) and the notion of generic semistability and the invariant μ for objects in it. (Attention: We do not develop the general theory of rigid analytic stacks nor log rigid spaces. We only define the above categories.) An object in these categories does not contain an information on isocrystals nor lattices. Then, in the case of curves, we can rewrite the equivalences (0.18), (0.19) and (0.20) further to obtain the equivalences

$$(0.21) \quad \text{Rep}_{K^\sigma}(\pi_1^t(X)) \xrightarrow{=} \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} F\text{-Vect}([\overline{\mathcal{Y}}/G_Y]_K)^{\text{gss}, \mu=0},$$

$$(0.22) \quad \text{Rep}_{K^\sigma}(\pi_1^t(X)) \xrightarrow{=} \varinjlim_{(n,p)=1} F\text{-Vect}((\overline{\mathcal{X}}, \mathcal{Z})_K^{1/n})^{\text{gss}, \mu=0},$$

$$(0.23) \quad \text{Rep}_{K^\sigma}(\pi_1^t(X)) \xrightarrow{=} \text{Par-}F\text{-Vect}((\overline{\mathcal{X}}, \mathcal{Z})_K)^{\text{gss}, \mu=0},$$

which will be further better p -adic analogues of (0.2), (0.3), (0.5). These equivalences are essentially conjectured by Weng [51] as a micro reciprocity law in log rigid analytic geometry.

In the case of p -torsion coefficient, Ogus-Vologodsky [36] and Gros-Le Stum-Quirós [12] prove the Simpson correspondence between the category of integrable connections and Higgs bundles, and the logarithmic version of it is proved by Scheppler [38]. Our results are different from theirs because we treat p -adic coefficient. On the other hand, we have to say that our results are not fully developed in the sense that the Higgs bundle does not appear in our equivalence. We expect that certain generalization of the results of Scheppler (to the p -adic coefficient case) is related to certain generalization (to the Higgs case) of our result.

The content of each section is as follows: In the first section, we review the definition and some results concerning certain properties on log- ∇ -modules and isocrystals which we developed in [44]. Note that we also add some results which were not proved there but useful in this paper. In the second section, we give a definition of the category of (log) convergent isocrystals on (fine log) algebraic stacks and prove the equivalences (0.7), (0.8) and (0.9). We also relate the variants of right hand sides of (0.8) and (0.9) without Frobenius structures, with log structures and with exponent conditions in the case of curves. In the third section, we introduce the category of semisimply adjusted parabolic unit-root log convergent F -isocrystals

and prove the equivalence (0.10). In the course of the proof, we prove the equivalence of the variants of right hand sides of (0.9) and (0.10) without Frobenius structures, with log structures and with exponent conditions. In the fourth section, we work in the lifted situation and prove the equivalences (0.11), (0.12) and (0.13). We also prove a comparison result between the category of vector bundles on certain ind-stacks and the category of parabolic vector bundles on formal schemes which is a formal version of the results of Iyer-Simpson [14] and Borne [3] [4]. In the fifth section, we introduce the notion of generic semistability and the invariant μ for objects in several categories and prove the equivalences (0.15)–(0.23), using results of Katz [18] and Crew [8], [9].

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Convention

Throughout this paper, p is a fixed prime number and q is a fixed power of p . K is a complete discrete valuation field of characteristic zero with ring of integers O_K and perfect residue field k containing \mathbb{F}_q . The maximal ideal of O_K is denoted by \mathfrak{m}_K . We fix a valuation $|\cdot| : K \rightarrow \mathbb{R}_{\geq 0}$ induced by the discrete valuation on K and let us put $\Gamma^* := \sqrt{|K^\times|} \cup \{0\} \subseteq \mathbb{R}_{\geq 0}$. Assume moreover that there exists an endomorphism $\sigma : K \rightarrow K$ inducing the endomorphism $O_K \rightarrow O_K$ (denoted also by σ) which lifts the q -th power map on k . Let K^σ be the fixed field of σ and let $O_K^\sigma := K^\sigma \cap O_K$.

The category of schemes separated of finite type over k is denoted by Sch . Following [21], a variety over k (or a k -variety) means an object in Sch which is reduced. For $X \in \text{Sch}$ with X connected, let $\text{Rep}_{K^\sigma}(\pi_1(X))$ be the category of finite dimensional continuous representations of the fundamental group $\pi_1(X)$ of X over K^σ and let $\text{Rep}_{O_K^\sigma}(\pi_1(X))$ be the category of continuous representations of the fundamental group $\pi_1(X)$ of X to free O_K^σ -modules of finite rank. For $X \in \text{Sch}$, let $\text{Sm}_{K^\sigma}(X)$ be the category of smooth K^σ -sheaves on X_{et} and let $\text{Sm}_{O_K^\sigma}(X)$ be the category of smooth O_K^σ -sheaves on X_{et} . We have the well-known equivalences $\text{Rep}_{K^\sigma}(\pi_1(X)) \cong \text{Sm}_{K^\sigma}(X)$, $\text{Rep}_{O_K^\sigma}(\pi_1(X)) \cong \text{Sm}_{O_K^\sigma}(X)$ for $X \in \text{Sch}$ with X connected. For a p -adic formal scheme \mathcal{X} separated of finite type over $\text{Spf } O_K$, we define the categories $\text{Sm}_{K^\sigma}(\mathcal{X})$, $\text{Sm}_{O_K^\sigma}(\mathcal{X})$ in the same way.

For $X \in \text{Sch}$, we denote the category of convergent isocrystals (resp. convergent F -isocrystals, unit-root convergent F -isocrystals) on X over K by $\text{Isoc}(X)$ (resp. $F\text{-Isoc}(X)$, $F\text{-Isoc}(X)^\circ$). (For precise definition and basic properties, see

[35], [1], [24] and [9]. For the definition of $F\text{-Isoc}(X)$ and $F\text{-Isoc}(X)^\circ$, we follow the definition in [9], not that in [35].) For an open immersion $X \hookrightarrow \overline{X}$ in Sch , we denote the category of overconvergent isocrystals (resp. overconvergent F -isocrystals, unit-root overconvergent F -isocrystals) on (X, \overline{X}) over K by $\text{Isoc}^\dagger(X, \overline{X})$ (resp. $F\text{-Isoc}^\dagger(X, \overline{X}), F\text{-Isoc}^\dagger(X, \overline{X})^\circ$). (For precise definition and basic properties, see [1], [24], [9] and [21].) Let LSch be the category of fine log schemes separated of finite type over k . For $(X, M_X) \in \text{LSch}$, the category of locally free log convergent isocrystals on (X, M_X) over K (called locally free isocrystals on the log convergent site $((X, M_X)/\text{Spf } O_K)_{\text{conv}}$ in [44], [46]) by $\text{Isoc}^{\log}(X, M_X)$. The category of locally free log convergent F -isocrystals on (X, M_X) over K (that is, the category of pairs (\mathcal{E}, Ψ) consisting of $\mathcal{E} \in \text{Isoc}^{\log}(X, M_X)$ and an isomorphism $\Psi : F^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$, where F is the σ -linear endofunctor on $\text{Isoc}^{\log}(X, M_X)$ induced by q -th power map on (X, M_X) and σ) by $F\text{-Isoc}^{\log}(X, M_X)$. (For precise definition and basic properties, see [21], [46]. See also [41], [42].)

For a functor $\Phi : \mathcal{C} \longrightarrow \text{Sch}$, we define the category $\text{Sm}_{K^\sigma}(\Phi)$ of smooth K^σ -sheaves on Φ (resp. the category $\text{Sm}_{O_K^\sigma}(\Phi)$ of smooth O_K^σ -sheaves on Φ) as the category of pairs

$$(\{\mathcal{E}_Y\}_{Y \in \text{Ob}(\mathcal{C})}, \{\varphi_{\mathcal{E}}\}_{\varphi: Y \rightarrow Y' \in \text{Mor}(\mathcal{C})}),$$

where $\mathcal{E}_Y \in \text{Sm}_{K^\sigma}(Y)$ (resp. $\mathcal{E}_Y \in \text{Sm}_{O_K^\sigma}(Y)$) and $\varphi_{\mathcal{E}}$ is an isomorphism $\Phi(\varphi)^*\mathcal{E}_{Y'} \xrightarrow{\sim} \mathcal{E}_Y$ in $\text{Sm}_{K^\sigma}(Y)$ (resp. $\text{Sm}_{O_K^\sigma}(Y)$) satisfying the cocycle condition $\varphi_{\mathcal{E}} \circ \Phi(\varphi)^*\varphi'_{\mathcal{E}} = (\varphi' \circ \varphi)_{\mathcal{E}}$ for $Y \xrightarrow{\varphi} Y' \xrightarrow{\varphi'} Y''$ in \mathcal{C} . For a functor $\Phi : \mathcal{C} \longrightarrow \text{Sch}$, we define the category $\text{Isoc}(\Phi)$ of convergent isocrystals on Φ over K as the category of pairs

$$(\{\mathcal{E}_Y\}_{Y \in \text{Ob}(\mathcal{C})}, \{\varphi_{\mathcal{E}}\}_{\varphi: Y \rightarrow Y' \in \text{Mor}(\mathcal{C})}),$$

where $\mathcal{E}_Y \in \text{Isoc}(\Phi(Y))$ and $\varphi_{\mathcal{E}}$ is an isomorphism $\Phi(\varphi)^*\mathcal{E}_{Y'} \xrightarrow{\sim} \mathcal{E}_Y$ in $\text{Isoc}(\Phi(Y))$ satisfying the cocycle condition $\varphi_{\mathcal{E}} \circ \Phi(\varphi)^*\varphi'_{\mathcal{E}} = (\varphi' \circ \varphi)_{\mathcal{E}}$ for $Y \xrightarrow{\varphi} Y' \xrightarrow{\varphi'} Y''$ in \mathcal{C} . We can also define the category $F\text{-Isoc}(\Phi)$ of convergent F -isocrystals on Φ over K and the category $F\text{-Isoc}(\Phi)^\circ$ of unit-root convergent F -isocrystals on Φ over K in the same way. Similarly, for a functor $\Phi : \mathcal{C} \longrightarrow (\text{open immersions in Sch})$, we can define the category $\text{Isoc}^\dagger(\Phi)$ of overconvergent isocrystals on Φ over K , the category $F\text{-Isoc}^\dagger(\Phi)$ of overconvergent F -isocrystals on Φ over K and the category $F\text{-Isoc}^\dagger(\Phi)^\circ$ of unit-root overconvergent F -isocrystals on Φ over K . Also, for a functor $\Phi : \mathcal{C} \longrightarrow \text{LSch}$, we can define the category $\text{Isoc}^{\log}(\Phi)$ of locally free log convergent isocrystals on Φ over K and the category $F\text{-Isoc}^{\log}(\Phi)$ of locally free log convergent F -isocrystals on Φ over K .

Note that a diagram of schemes X_\bullet in Sch can be regarded as a functor $\mathcal{C} \longrightarrow \text{Sch}$ as above. So we can define the categories $\text{Sm}_{K^\sigma}(X_\bullet), \text{Sm}_{O_K^\sigma}(X_\bullet), \text{Isoc}(X_\bullet), F\text{-Isoc}(X_\bullet), F\text{-Isoc}(X_\bullet)^\circ$ in the above way. Also, for a diagram $X_\bullet \hookrightarrow \overline{X}_\bullet$ of open immersions in Sch , we can define the categories $\text{Isoc}^\dagger(X_\bullet, \overline{X}_\bullet), F\text{-Isoc}^\dagger(X_\bullet, \overline{X}_\bullet), F\text{-Isoc}^\dagger(X_\bullet, \overline{X}_\bullet)^\circ$ and for a diagram $(X_\bullet, M_{X_\bullet})$ in LSch , we can define the categories $\text{Isoc}^{\log}(X_\bullet, M_{X_\bullet}), F\text{-Isoc}^{\log}(X_\bullet, M_{X_\bullet})$.

For a p -adic formal scheme \mathcal{X} separated of finite type over $\mathrm{Spf} O_K$, we denote the associated rigid space over K by \mathcal{X}_K . For a fine log structure M on a (formal) scheme X , we put $\overline{M} := M/\mathcal{O}_X^\times$. A morphism of log (formal) schemes $f : (X, M_X) \longrightarrow (Y, M_Y)$ is called strict when $f^*M_Y = M_X$. When X is a smooth scheme over O_K/\mathfrak{m}_K^a for some a or a p -adic formal scheme smooth over $\mathrm{Spf} O_K$ and Z is a relative simple normal crossing divisor on X , we denote by (X, Z) the fine log (formal) scheme with underlying (formal) scheme X whose log structure is associated to Z .

For subsets Σ, Σ' of the form $\Sigma = \prod_{i=1}^r \Sigma_i, \Sigma' = \prod_{i=1}^r \Sigma'_i$ in \mathbb{Z}_p^r and $n \in \mathbb{N}, a = (a_i)_{1 \leq i \leq r} \in \mathbb{N}^r$, we define $\Sigma + \Sigma', n\Sigma, a\Sigma$ by

$$\begin{aligned} \Sigma + \Sigma' &:= \prod_{i=1}^r \{\xi_i + \xi'_i \mid \xi_i \in \Sigma_i, \xi'_i \in \Sigma'_i\}, \\ n\Sigma &:= \prod_{i=1}^r \{n\xi_i \mid \xi_i \in \Sigma_i\}, \quad a\Sigma := \prod_{i=1}^r \{a_i \xi_i \mid \xi_i \in \Sigma_i\}. \end{aligned}$$

Also, the set $\{0\}^r$ in \mathbb{Z}_p^r is denoted simply by $\mathbf{0}$.

Finally, a discrete valuation always means a discrete valuation of rank one.

1 Log- ∇ -modules and isocrystals

In this section, we review the definition and some results concerning certain properties on log- ∇ -modules and isocrystals which we developed in [44] (which generalizes some results in [21]). We also add some more terminologies and results which were not treated there but useful in this paper.

1.1 Log- ∇ -modules

Let L be a field containing K complete with respect to a multiplicative norm (also denoted by $|\cdot|$) which extends the given absolute value on K . For a morphism $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$ of rigid spaces over L , a ∇ -module of on \mathfrak{X} relative to \mathfrak{Y} is defined to be a pair (E, ∇) consisting of a coherent module E on \mathfrak{X} endowed with an integrable $f^{-1}\mathcal{O}_{\mathfrak{Y}}$ -linear connection $\nabla : E \longrightarrow E \otimes_{\mathcal{O}_{\mathfrak{X}}} \Omega_{\mathfrak{X}/\mathfrak{Y}}^1$. In the case $\mathfrak{Y} = \mathrm{Spm} L$, we omit the term ‘relative to \mathfrak{Y} ’. We denote the category of ∇ -modules on \mathfrak{X} by $\mathrm{NM}_{\mathfrak{X}}$.

For a morphism $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$ of rigid spaces over L and elements x_1, \dots, x_r in $\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$, a log- ∇ -module on \mathfrak{X} with respect to x_1, \dots, x_r relative to \mathfrak{Y} is defined to be a pair (E, ∇) consisting of a locally free module of finite rank E on \mathfrak{X} endowed with an integrable $f^{-1}\mathcal{O}_{\mathfrak{Y}}$ -linear log connection $\nabla : E \longrightarrow E \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega_{\mathfrak{X}/\mathfrak{Y}}^1$. (Here $\omega_{\mathfrak{X}/\mathfrak{Y}}^1$ is defined by

$$(1.1) \quad \omega_{\mathfrak{X}/\mathfrak{Y}}^1 := (\Omega_{\mathfrak{X}/\mathfrak{Y}}^1 \oplus \bigoplus_{i=1}^r \mathcal{O}_{\mathfrak{X}} \cdot d\log x_i) / N,$$

where N is the sheaf locally generated by $(dx_i, 0) - (0, x_i \mathrm{dlog} x_i)$ ($1 \leq i \leq r$). In the case $\mathfrak{Y} = \mathrm{Spm} L$, we omit the term ‘relative to \mathfrak{Y} ’. We denote the category of $\log\text{-}\nabla$ -modules on \mathfrak{X} with respect to x_1, \dots, x_r by $\mathrm{LNM}_{\mathfrak{X}}$.

Remark 1.1. When \mathcal{X} is a p -adic formal scheme smooth separated of finite type over $\mathrm{Spf} O_K$ and $\mathcal{Z} = \bigcup_{i=1}^r \mathcal{Z}_i$ is a relative simple normal crossing divisor on \mathcal{X} (with each \mathcal{Z}_i irreducible), we have sections $x_1, \dots, x_r \in \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ which cut out $\mathcal{Z}_1, \dots, \mathcal{Z}_r$ Zariski locally on \mathcal{X} . So, locally on \mathcal{X} , we can define the notion of $\log\text{-}\nabla$ -modules on \mathcal{X}_K with respect to x_1, \dots, x_r (relative to $\mathrm{Spm} K$). In this case, we see that this definition is independent of the choice of x_i ’s above, because $\omega_{\mathcal{X}_K/\mathrm{Spm} K}^1$ defined in (1.1) is nothing but the coherent sheaf on \mathcal{X}_K induced by the log differential module $\Omega_{\mathcal{X}}^1(\log \mathcal{Z})$ on the formal scheme \mathcal{X} . So, in this case, we can define the notion of $\log\text{-}\nabla$ -module on $(\mathcal{X}_K, \mathcal{Z}_K)$ globally, by patching the above local definition of $\log\text{-}\nabla$ -modules on \mathcal{X} with respect to x_1, \dots, x_r . We denote the category of $\log\text{-}\nabla$ -modules on $(\mathcal{X}_K, \mathcal{Z}_K)$ by $\mathrm{LNM}_{(\mathcal{X}_K, \mathcal{Z}_K)}$ and also by $\mathrm{LNM}_{\mathcal{X}_K}$ when there will be no confusion on \mathcal{Z}_K .

Next, let \mathfrak{X} be a smooth rigid space over L endowed with sections $x_1, \dots, x_r \in \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ whose zero loci are affinoid, smooth and meet transversally. Let us put $\mathfrak{D}_i := \{x_i = 0\}$ and $M_i := \mathrm{Im}(\Omega_{\mathfrak{X}/K}^1 \oplus \bigoplus_{j \neq i} \mathcal{O}_{\mathfrak{X}} \mathrm{dlog} x_j \longrightarrow \omega_{\mathfrak{X}}^1)$. Then the composite map

$$E \xrightarrow{\nabla} E \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega_{\mathfrak{X}}^1 \longrightarrow E \otimes_{\mathcal{O}_{\mathfrak{X}}} (\omega_{\mathfrak{X}}^1/M_i) \cong E|_{\mathfrak{D}_i} \mathrm{dlog} x_i \cong E|_{\mathfrak{D}_i}$$

naturally induces an element res_i in $\mathrm{End}_{\mathcal{O}_{\mathfrak{D}_i}}(E|_{\mathfrak{D}_i})$, which we call the residue of (E, ∇) along \mathfrak{D}_i . By [45, 1.24], we can take the minimal monic polynomial $P_i(x) \in K[x]$ satisfying $P_i(\mathrm{res}_i) = 0$. We call the roots of $P_i(x)$ the exponents of (E, ∇) along \mathfrak{D}_i . For $\mathfrak{X}, x_1, \dots, x_r, \mathfrak{D}_1, \dots, \mathfrak{D}_r$ as above and $\Sigma := \prod_{i=1}^r \Sigma_i \subseteq \mathbb{Z}_p^r$, we denote the category of $\log\text{-}\nabla$ -modules on \mathfrak{X} with respect to x_1, \dots, x_r whose exponents along \mathfrak{D}_i are contained in Σ_i ($1 \leq i \leq r$) by $\mathrm{LNM}_{\mathfrak{X}, \Sigma}$. In the situation of Remark 1.1, the category $\mathrm{LNM}_{(\mathcal{X}_K, \mathcal{Z}_K), \Sigma}$ is defined by patching this definition.

We call an interval I in $[0, \infty)$ aligned if any endpoint of I at which it is closed is contained in Γ^* . We call an interval I in $[0, \infty)$ quasi-open if it is open at non-zero endpoints. For an aligned interval I , we define the rigid space $A_L^n(I)$ by $A_L^n(I) := \{(t_1, \dots, t_n) \in \mathbb{A}_L^{n, \mathrm{an}} \mid \forall i, |t_i| \in I\}$.

Following [21, 3.2.4], we use the following convention: For a smooth affinoid rigid space \mathfrak{X} over L we put $\omega_{\mathfrak{X} \times A_L^n[0, 0]}^1 := \Omega_{\mathfrak{X}}^1 \oplus \bigoplus_{i=1}^n \mathcal{O}_{\mathfrak{X}} \mathrm{dlog} t_i$, where $\mathrm{dlog} t_i$ is the free generator ‘corresponding to the i -th coordinate of $A_L^n[0, 0]$ ’. Using this, we can define the notion of a $\log\text{-}\nabla$ -module (E, ∇) on $\mathfrak{X} \times A_L^n[0, 0]$ with respect to t_1, \dots, t_n and the notion of the residue, the exponents of (E, ∇) along $\{t_i = 0\}$ in natural way: To give a $\log\text{-}\nabla$ -module on $\mathfrak{X} \times A_L^n[0, 0]$ with respect to t_1, \dots, t_n is equivalent to give a ∇ -module (E, ∇) on \mathfrak{X} and commuting endomorphisms $\partial_i := t_i \frac{\partial}{\partial t_i}$ of (E, ∇) ($1 \leq i \leq n$). Also, we can define the category $\mathrm{LNM}_{\mathfrak{X} \times A_L^n[0, 0], \Sigma}$ for $\Sigma = \prod_{i=1}^n \Sigma_i \subseteq \mathbb{Z}_p^n$ as above: A $\log\text{-}\nabla$ -module on $\mathfrak{X} \times A_L^n[0, 0]$ with respect to t_1, \dots, t_n , regarded as

a ∇ -module (E, ∇) on \mathfrak{X} endowed with commuting endomorphisms $\partial_i := t_i \frac{\partial}{\partial t_i}$ ($1 \leq i \leq n$), is in the category $\text{LNM}_{\mathfrak{X} \times A_L^n[0,0], \Sigma}$ if and only if all the eigenvalues of ∂_i are in Σ_i ($1 \leq i \leq n$).

For an aligned interval $I \subseteq [0, \infty)$ and $\xi := (\xi_1, \dots, \xi_n) \in \mathbb{Z}_p^n$, we define the log- ∇ -module (M_ξ, ∇_{M_ξ}) on $A_L^n(I)$ with respect to t_1, \dots, t_n (which are the coordinates) as the log- ∇ -module $(\mathcal{O}_{A_L^n(I)}, d + \sum_{i=1}^n \xi_i d \log t_i)$. Following [44, 1.3] (cf. [21, 3.2.5]), we define the notion of Σ -constance and Σ -unipotence of log- ∇ -modules as follows (we also introduce the notion of Σ -semisimplicity):

Definition 1.2. *Let \mathfrak{X} be a smooth rigid space over L . Let $I \subseteq [0, \infty)$ be an aligned interval and fix $\Sigma := \prod_{i=1}^n \Sigma_i \subseteq \mathbb{Z}_p^n$.*

(1) *An object (E, ∇) in $\text{LNM}_{\mathfrak{X} \times A_L^n(I), \Sigma}$ ($\mathfrak{X} \times A_L^n(I)$ is endowed with t_1, \dots, t_n , where t_i 's are the coordinates in $A_L^n(I)$) is called Σ -constant if (E, ∇) has the form $\pi_1^*(F, \nabla_F) \otimes \pi_2^*(M_\xi, \nabla_{M_\xi})$ for some ∇ -module (F, ∇_F) on \mathfrak{X} and $\xi \in \Sigma$, where $\pi_1 : \mathfrak{X} \times A_L^n(I) \rightarrow \mathfrak{X}$, $\pi_2 : \mathfrak{X} \times A_L^n(I) \rightarrow A_L^n(I)$ denote the projections. An object in $\text{LNM}_{\mathfrak{X} \times A_L^n(I), \Sigma}$ is called Σ -semisimple if it is a direct sum of Σ -constant ones.*

(2) *An object (E, ∇) in $\text{LNM}_{\mathfrak{X} \times A_L^n(I), \Sigma}$ is called Σ -unipotent if (E, ∇) admits a filtration*

$$0 = E_0 \subset E_1 \subset \dots \subset E_m = E$$

by sub log- ∇ -modules whose successive quotients are Σ -constant log- ∇ -modules.

We denote the category of Σ -semisimple (Σ -unipotent) log- ∇ -modules on $\mathfrak{X} \times A_L^n(I)$ with respect to t_1, \dots, t_n by $\text{SLNM}_{\mathfrak{X} \times A_L^n(I), \Sigma}$ ($\text{ULNM}_{\mathfrak{X} \times A_L^n(I), \Sigma}$). Note that we have $\text{SLNM}_{\mathfrak{X} \times A_L^n(I), \Sigma} \subseteq \text{ULNM}_{\mathfrak{X} \times A_L^n(I), \Sigma}$.

Here we give a remark which is the same as [44, 1.4]: When I does not contain 0, the log- ∇ -modules M_ξ and $M_{\xi'}$ ($\xi, \xi' \in \Sigma$) are isomorphic if $\xi - \xi'$ is contained in \mathbb{Z}^n . So we see that the notion of Σ -semisimplicity and Σ -unipotence only depends on the image $\overline{\Sigma}$ of Σ in $\mathbb{Z}_p^n / \mathbb{Z}^n$ in the following sense: An object (E, ∇) in $\text{LNM}_{\mathfrak{X} \times A_L^n(I), \Sigma}$ is Σ -semisimple (Σ -unipotent) if and only if it is $\tau(\overline{\Sigma})$ -semisimple ($\tau(\overline{\Sigma})$ -unipotent) for some (or any) section $\tau : \mathbb{Z}_p^n / \mathbb{Z}^n \rightarrow \mathbb{Z}_p^n$ of the form $\tau = \prod_{i=1}^n \tau_i$ of the canonical projection $\mathbb{Z}_p^n \rightarrow \mathbb{Z}_p^n / \mathbb{Z}^n$. So, in this case, we will say also that (E, ∇) is $\overline{\Sigma}$ -semisimple ($\overline{\Sigma}$ -unipotent), by abuse of terminology.

Before giving properties on Σ -semisimple and Σ -unipotent log- ∇ -modules, we recall (and introduce) several terminologies on subsets in \mathbb{Z}_p^r . (Some of them are used not in this subsection but in later sections.) Recall that an element α in \mathbb{Z}_p is called p -adically non-Liouville if we have the equalities

$$\lim_{n \rightarrow \infty} |\alpha + n|^{1/n} = \lim_{n \rightarrow \infty} |\alpha - n|^{1/n} = 1.$$

Definition 1.3. (1) *A subset Σ in \mathbb{Z}_p is (NID) (resp. (NRD)) if, for any $\alpha, \beta \in \Sigma$, $\alpha - \beta$ is not a non-zero integer (resp. $\alpha - \beta$ is not a non-zero rational number).*

- (2) A subset Σ in \mathbb{Z}_p is (NLD) (resp. (SNLD)) if, for any $\alpha, \beta \in \Sigma$, $\alpha - \beta$ is a p -adically non-Liouville number (resp. for any $\alpha, \beta \in \Sigma$ and $a \in \mathbb{Z}_{(p)}$, $\alpha - \beta + a$ is a p -adically non-Liouville number).
- (3) A subset Σ in \mathbb{Z}_p^r of the form $\Sigma = \prod_{i=1}^r \Sigma_i$ is (NID) (resp. (NRD), (NLD), (SNLD)) if so is each Σ_i ($1 \leq i \leq r$).
- (4) A subset $\bar{\Sigma}$ in $\mathbb{Z}_p^r/\mathbb{Z}^r$ (resp. $\mathbb{Z}_p^r/\mathbb{Z}_{(p)}^r$) of the form $\bar{\Sigma} = \prod_{i=1}^r \bar{\Sigma}_i$ is (NLD) if, for any section $\tau : \mathbb{Z}_p^r/\mathbb{Z}^r \rightarrow \mathbb{Z}_p^r$ (resp. $\tau : \mathbb{Z}_p^r/\mathbb{Z}_{(p)}^r \rightarrow \mathbb{Z}_p^r$) of the natural projection $\mathbb{Z}_p^r \rightarrow \mathbb{Z}_p^r/\mathbb{Z}^r$ (resp. $\mathbb{Z}_p^r \rightarrow \mathbb{Z}_p^r/\mathbb{Z}_{(p)}^r$) of the form $\tau = \prod_{i=1}^r \tau_i$, $\tau(\bar{\Sigma})$ is (NLD). (Note that, when this condition is satisfied, $\tau(\bar{\Sigma})$ is (NID) and (NLD) (resp. (NRD) and (SNLD)). Note also that, if $\Sigma = \prod_{i=1}^r \Sigma_i \subseteq \mathbb{Z}_p^r$ is (NID) and (NLD) (resp. (NRD) and (SNLD)), the image $\bar{\Sigma}$ of Σ in $(\mathbb{Z}_p/\mathbb{Z})^r$ (resp. $\mathbb{Z}_p^r/\mathbb{Z}_{(p)}^r$) is (NLD) and we have $\Sigma = \tau(\bar{\Sigma})$ for some section τ as above.)

We have the following property, which we use in the later sections.

Lemma 1.4. *Let $\Sigma := \prod_{i=1}^r \Sigma_i$ be a subset in \mathbb{Z}_p^r which is (NRD) (resp. (SNLD)). Then, for any $m \in \mathbb{N}$ prime to p , $m\Sigma$ (see Convention for definition) is (NID) (resp. (NLD)).*

Proof. We only prove that $m\Sigma$ is (NLD) when Σ is (SNLD), because the other assertion is easy. We may assume that $r = 1$. Take any $\alpha, \beta \in \Sigma$ and put $\xi := m(\alpha - \beta)$. Then, for any $n \in \mathbb{N}$ with $n = mq + r$ ($q, r \in \mathbb{N}, 0 \leq r < m$), we have

$$\begin{aligned}
1 &\geq |\xi \pm n|^{1/n} = |m|^{1/n} \left| \left(\alpha - \beta \pm \frac{r}{m} \right) \pm q \right|^{\frac{1}{q} \cdot \frac{q}{n}} \\
&\geq \left| \left(\alpha - \beta \pm \frac{r}{m} \right) \pm q \right|^{\frac{1}{q} \cdot \frac{1}{m+1}} \\
&\geq \min_{a \in (-1, 1) \cap \frac{1}{m}\mathbb{Z}} (|(\alpha - \beta + a) \pm q|^{\frac{1}{q} \cdot \frac{1}{m+1}})
\end{aligned}$$

and the limit inferior of the right hand side as $n \rightarrow \infty$ is 1 since $(\alpha - \beta + a)$'s are p -adically non-Liouville by assumption. So we have $\underline{\lim}_{n \rightarrow \infty} |\xi \pm n|^{1/n} = 1$ and so $m\Sigma$ is (NLD). \square

Now let us recall the following proposition, which is partly proven in [44, 1.13, 1.15]:

Proposition 1.5. *Let $\Sigma = \prod_{i=1}^n \Sigma_i$ be a subset of \mathbb{Z}_p^n which is (NID) and (NLD). Then, for a smooth rigid space \mathfrak{X} over L and a quasi-open interval $I \subseteq [0, \infty)$, we have the equivalences of categories*

$$\begin{aligned}
\mathcal{U}_I : \text{LNM}_{\mathfrak{X} \times A_L^n[0,0], \Sigma} &= \text{ULNM}_{\mathfrak{X} \times A_L^n[0,0], \Sigma} \xrightarrow{=} \text{ULNM}_{\mathfrak{X} \times A_L^n(I), \Sigma}, \\
\mathcal{U}_I : \text{SLNM}_{\mathfrak{X} \times A_L^n[0,0], \Sigma} &\xrightarrow{=} \text{SLNM}_{\mathfrak{X} \times A_L^n(I), \Sigma}
\end{aligned}$$

defined as follows: An object $((E, \nabla), \{\partial_i\})$ in $\text{ULNM}_{\mathfrak{X} \times A_L^n[0,0], \Sigma}$ or $\text{SLNM}_{\mathfrak{X} \times A_L^n[0,0], \Sigma}$ (so (E, ∇) is a ∇ -module on \mathfrak{X}) is sent to π^*E (where π is the projection $\mathfrak{X} \times A_L^n(I) \rightarrow \mathfrak{X}$) endowed with the log connection

$$\mathbf{v} \mapsto \pi^*\nabla(\mathbf{v}) + \sum_{i=1}^n \pi^*(\partial_i)(\mathbf{v}) d\log t_i.$$

Proof. The first equivalence is a special case of [44, 1.15]. The essential surjectivity of the second functor follows from the definition of Σ -semisimplicity and the full faithfulness of the second functor follows from that of the first functor. \square

We also prove certain full faithfulness property of $\log\nabla$ -modules on relative annulus, which is a variant of [44, 1.14].

Proposition 1.6. *Let \mathfrak{X} be a smooth rigid space over L , let λ be an element in $(0, 1) \cap \Gamma^*$ and let $\Sigma_1 := \prod_{j=1}^n \Sigma_{1j}, \Sigma_2 := \prod_{j=1}^n \Sigma_{2j}$ be subsets of \mathbb{Z}_p^n such that for any j and for any $\xi_1 \in \Sigma_{1j}, \xi_2 \in \Sigma_{2j}$, we have $\xi_1 - \xi_2 \in \mathbb{Z}_p \setminus \mathbb{Z}_{<0}$. For $i = 1, 2$, let $E_i := (E_i, \nabla_i)$ be a Σ_i -unipotent $\log\nabla$ -module on $\mathfrak{X} \times A_L^n[0, 1)$ with respect to t_1, \dots, t_n (where t_1, \dots, t_n are the canonical coordinate of $A_L^n[0, 1)$) and let us put $E'_i := E_i|_{\mathfrak{X} \times A_K^n[\lambda, 1)}$, which is a Σ_i -unipotent $\log\nabla$ -module on $\mathfrak{X} \times A_L^n[\lambda, 1)$. Then the restriction functor induces the isomorphism*

$$\text{Hom}(E_1, E_2) \xrightarrow{=} \text{Hom}(E'_1, E'_2),$$

where Hom means the set of homomorphisms as $\log\nabla$ -modules.

Proof. The proof is analogous to that of [44, 1.14]. First, by Proposition 1.5, there exists a Σ_i -unipotent $\log\nabla$ -module F_i ($i = 1, 2$) on $\mathfrak{X} \times A_L^n[0, 0]$ with respect to t_1, \dots, t_n (which are ‘the coordinate of $A_L^n[0, 0]$ ’) such that $E_i = \mathcal{U}_{[0,1)}(F_i)$, and we have $\text{Hom}(F_1, F_2) \xrightarrow{=} \text{Hom}(E_1, E_2)$, $E'_i = \mathcal{U}_{[\lambda,1)}(F_i)$. So it suffices to prove that the functor $\mathcal{U}_{[\lambda,1)}$ induces the isomorphism

$$\text{Hom}(F_1, F_2) \xrightarrow{=} \text{Hom}(\mathcal{U}_{[\lambda,1)}(F_1), \mathcal{U}_{[\lambda,1)}(F_2)).$$

In the following, for a $\log\nabla$ -module N on $\mathfrak{X} \times A_L^n(J)$ ($J = [0, 0]$ or $[\lambda, 1)$) with respect to t_1, \dots, t_n , we denote the associated $\log\nabla$ -module on $\mathfrak{X} \times A_L^n(J)$ with respect to t_1, \dots, t_n relative to \mathfrak{X} by \overline{N} .

Let us put $F = F_1^\vee \otimes F_2$ (as $\log\nabla$ -module). Then it suffices to prove that the map

$$H^a(\mathfrak{X}, F \otimes \omega_{\mathfrak{X} \times A_L^n[0,0]}^\bullet) \longrightarrow H^a(\mathfrak{X} \times A_L^n[\lambda, 1), \mathcal{U}_{[\lambda,1)}(F) \otimes \omega_{\mathfrak{X} \times A_L^n[\lambda,1)}^\bullet)$$

is an isomorphism for $a = 0$ and injective for $a = 1$. We may assume that \mathfrak{X} is affinoid by considering the spectral sequence induced by admissible hypercovering by affinoids. By the same technique and the five lemma, we may assume that \mathfrak{X}

is affinoid and that F has the form $F_0 \otimes \pi^* M_\xi$ (where π is ‘the projection $\mathfrak{X} \times A_L^n[0, 0] \rightarrow A_L^n[0, 0]$ ’) for some $\xi := \xi_2 - \xi_1$ ($\xi_i \in \Sigma_i$) and for some ∇ -module F_0 on \mathfrak{X} with F_0 free as $\mathcal{O}_{\mathfrak{X}}$ -module.

By considering the Katz-Oda type spectral sequence for the diagram $\mathfrak{X} \times A_L^n(J) \rightarrow \mathfrak{X} \rightarrow \mathrm{Spm} K$ for $J = [0, 0]$ or $[\lambda, 1]$, we obtain the spectral sequence

$$\begin{aligned} E_2^{a,b} &= H^a(\Gamma(\mathfrak{X}, \omega_{\mathfrak{X}/L}^\bullet) \otimes H^b(\mathfrak{X} \times A_L^n(J), \overline{F_J} \otimes \omega_{\mathfrak{X} \times A_L^n(J)/L}^\bullet)) \\ &\implies H^{a+b}(\mathfrak{X} \times A_L^n(J), F_J \otimes \omega_{\mathfrak{X} \times A_L^n(J)/L}^\bullet), \end{aligned}$$

where $F_J = F$ (resp. $\mathcal{U}_{[\lambda, 1]}(F)$) when $J = [0, 0]$ (resp. $J = [\lambda, 1]$). From this, we see that it suffices to prove the map

$$(1.2) \quad H^a(\mathfrak{X}, \overline{F} \otimes \omega_{\mathfrak{X} \times A_L^n[0, 0]/\mathfrak{X}}^\bullet) \rightarrow H^a(\mathfrak{X} \times A_L^n[\lambda, 1], \overline{\mathcal{U}_{[\lambda, 1]}(F)} \otimes \omega_{\mathfrak{X} \times A_L^n[\lambda, 1]/\mathfrak{X}}^\bullet)$$

induced by $\mathcal{U}_{[\lambda, 1]}$ is an isomorphism for $a = 0$ and injective for $a = 1$.

Since \overline{F} is a finite direct sum of $\pi^* M_\xi$, we may assume that $\overline{F} = \pi^* M_\xi$. If we put $\xi = (\eta_j)_{j=1}^n$, we have $\eta_j \in \mathbb{Z}_p \setminus \mathbb{Z}_{>0}$ for all j by assumption on Σ_i 's. Then, when $a = 0$, the both hand sides on (1.2) are equal to 0 if there exists some j with $\eta_j \in \mathbb{Z}_p \setminus \mathbb{Z}$ and equal to $\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \prod_{j=1}^n t_j^{-\eta_j}$ if $\eta_j \in \mathbb{Z}_{\leq 0}$ for all j . Hence they are isomorphic when $a = 0$.

For general a , the left hand side (resp. the right hand side) of (1.2) (in the case $\overline{F} = \pi^* M_\xi$) is the a -th cohomology of the left hand side (resp. the right hand side) of the following map of complexes which is induced by $\mathcal{U}_{[\lambda, 1]}$:

$$(1.3) \quad \Gamma(\mathfrak{X}, \overline{\pi^* M_\xi} \otimes \omega_{\mathfrak{X} \times A_L^n[0, 0]/\mathfrak{X}}^\bullet) \rightarrow \Gamma(\mathfrak{X} \times A_L^n[\lambda, 1], \overline{\mathcal{U}_{[\lambda, 1]}(\pi^* M_\xi)} \otimes \omega_{\mathfrak{X} \times A_L^n[\lambda, 1]/\mathfrak{X}}^\bullet).$$

(Here the complex is defined as the log de Rham complex associated to the log- ∇ -module structure on $\overline{\pi^* M_\xi}$ and $\overline{\mathcal{U}_{[\lambda, 1]}(\pi^* M_\xi)}$.) Since the map

$$\Gamma(\mathfrak{X} \times A_L^n[\lambda, 1], \overline{\mathcal{U}_{[\lambda, 1]}(\pi^* M_\xi)} \otimes \omega_{\mathfrak{X} \times A_L^n[\lambda, 1]/\mathfrak{X}}^\bullet) \rightarrow \Gamma(\mathfrak{X}, \overline{\pi^* M_\xi} \otimes \omega_{\mathfrak{X} \times A_L^n[0, 0]/\mathfrak{X}}^\bullet)$$

of ‘taking the constant coefficient’ gives a left inverse of (1.2), we see that the map (1.3) induces injection on cohomologies. So the map (1.2) is injective for any a and the proof is finished. \square

1.2 Isocrystals

In this subsection, we review the definition of certain properties on isocrystals which we introduced in [44] and recall some results proven there. We also add some more new terminologies and results which are useful in later sections.

To do this, first we recall terminologies on frames.

Definition 1.7 ([21, 2.2.4, 4.2.1]). (1) *A frame (or an affine frame) is a triple $(X, \overline{X}, \overline{\mathcal{X}})$ consisting of k -varieties X, \overline{X} and a p -adic affine formal scheme $\overline{\mathcal{X}}$*

of finite type over $\mathrm{Spf} O_K$ endowed with a closed immersion $i : \overline{X} \hookrightarrow \overline{\mathcal{X}}$ over $\mathrm{Spf} O_K$ and an open immersion $j : X \hookrightarrow \overline{X}$ over k such that $\overline{\mathcal{X}}$ is formally smooth over O_K on a neighborhood of X . We say that the frame encloses the pair (X, \overline{X}) .

- (2) A small frame is a frame $(X, \overline{X}, \overline{\mathcal{X}})$ such that \overline{X} is isomorphic (via i as in (1)) to $\overline{\mathcal{X}} \times_{\mathrm{Spf} O_K} \mathrm{Spec} k$ and that there exists an element $f \in \Gamma(\overline{\mathcal{X}}, \mathcal{O}_{\overline{\mathcal{X}}})$ with $X = \{f \neq 0\}$.

Remark 1.8. In [21], a frame is written as a tuple $(X, \overline{X}, \overline{\mathcal{X}}, i, j)$, but we denote it here simply as a triple $(X, \overline{X}, \overline{\mathcal{X}})$.

Definition 1.9 ([44, 3.3]). Let $X \hookrightarrow \overline{X}$ be an open immersion of smooth k -varieties such that $Z := \overline{X} \setminus X$ is a simple normal crossing divisor and let $Z = \bigcup_{i=1}^r Z_i$ be a decomposition of Z such that $Z = \bigcup_{\substack{1 \leq i \leq r \\ Z_i \neq \emptyset}} Z_i$ gives the decomposition of Z into irreducible components. A standard small frame enclosing (X, \overline{X}) is a small frame $\mathbf{X} := (X, \overline{X}, \overline{\mathcal{X}})$ enclosing (X, \overline{X}) which satisfies the following condition: There exist $t_1, \dots, t_r \in \Gamma(\overline{\mathcal{X}}, \mathcal{O}_{\overline{\mathcal{X}}})$ such that, if we denote the zero locus of t_i in $\overline{\mathcal{X}}$ by \mathcal{Z}_i , each \mathcal{Z}_i is irreducible (possibly empty) and that $\mathcal{Z} = \bigcup_{i=1}^r \mathcal{Z}_i$ (which we call the lift of Z) is a relative simple normal crossing divisor of $\overline{\mathcal{X}}$ satisfying $Z_i = \mathcal{Z}_i \times_{\overline{\mathcal{X}}} \overline{X}$. We call a pair $(\mathbf{X}, (t_1, \dots, t_r))$ a charted standard small frame. When $r = 1$, we call \mathbf{X} a smooth standard small frame and the pair (\mathbf{X}, t_1) a charted smooth standard small frame.

We also introduce the notion of charted smooth standard small frame with generic point as follows (It is essentially already appeared in the paper [44], but it is convenient to give it a name):

Definition 1.10. Let $X \hookrightarrow \overline{X}$ be an open immersion of smooth k -varieties such that $Z := \overline{X} \setminus X$ is an irreducible smooth divisor. A charted smooth standard small frame with generic point enclosing (X, \overline{X}) is a triple (\mathbf{X}, t_1, L) consisting of a charted smooth standard small frame $(\mathbf{X}, t_1) = ((X, \overline{X}, \overline{\mathcal{X}}), t_1)$ and an injection $\Gamma(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}}) \hookrightarrow L$ (where $\mathcal{Z} := \{t_1 = 0\}$ is the lift of Z) into a field L endowed with a complete multiplicative norm which restricts to the supremum norm on $\Gamma(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}})$.

We introduce the notion of a morphism of charted smooth standard small frames (with generic points) as follows:

Definition 1.11. A morphism $f : ((X', \overline{X}', \overline{\mathcal{X}}'), t') \longrightarrow ((X, \overline{X}, \overline{\mathcal{X}}), t)$ of charted smooth standard small frames is morphisms $X' \longrightarrow X, \overline{X}' \longrightarrow \overline{X}, \overline{\mathcal{X}}' \xrightarrow{\tilde{f}} \overline{\mathcal{X}}$ compatible with the structure of smooth standard small frames such that $\tilde{f}^* t = t'^n$ for some positive integer n . A morphism $((X', \overline{X}', \overline{\mathcal{X}}'), t', L') \longrightarrow ((X, \overline{X}, \overline{\mathcal{X}}), t, L)$ of charted smooth standard small frames with generic points is a morphism $f :$

$((X', \overline{X}', \overline{\mathcal{X}}'), t') \longrightarrow ((X, \overline{X}, \overline{\mathcal{X}}), t)$ of charted smooth standard small frames endowed with a continuous morphism $g : L \longrightarrow L'$ of fields such that, if we put $\mathcal{Z} := \{t = 0\}$, $\mathcal{Z}' := \{t' = 0\}$, the diagram

$$\begin{array}{ccc} \Gamma(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}}) & \xrightarrow{f^*} & \Gamma(\mathcal{Z}', \mathcal{O}_{\mathcal{Z}'}) \\ \cap \downarrow & & \cap \downarrow \\ L & \xrightarrow{g} & L' \end{array}$$

is commutative, where f^* is the homomorphism induced by f .

As for the existence of a morphism of charted smooth standard small frames (with generic points), the following lemma will be useful later.

Lemma 1.12. *Let $X \hookrightarrow \overline{X}$ ($X' \hookrightarrow \overline{X}'$) be an open immersion of affine smooth k -varieties such that $Z = \overline{X} \setminus X$ ($Z' = \overline{X}' \setminus X'$) is an irreducible smooth divisor defined as the zero section of $\bar{t} \in \Gamma(\overline{X}, \mathcal{O}_{\overline{X}})$ ($\bar{t}' \in \Gamma(\overline{X}', \mathcal{O}_{\overline{X}'})$). Let $f : (\overline{X}', Z') \longrightarrow (\overline{X}, Z)$ be a log smooth morphism such that $f^*\bar{t} = \bar{t}'^n$ for some $n \in \mathbb{N}$ prime to p . Denote the morphism of pairs of schemes $(X', \overline{X}') \longrightarrow (X, \overline{X})$ induced by f also by f . Assume moreover that we are given a charted smooth standard small frame $((X, \overline{X}, \overline{\mathcal{X}}), t)$ enclosing (X, \overline{X}) such that $t \in \Gamma(\overline{\mathcal{X}}, \mathcal{O}_{\overline{\mathcal{X}}})$ is a lift of \bar{t} . Then there exists a charted smooth standard small frame $((X', \overline{X}', \overline{\mathcal{X}}'), t')$ enclosing (X', \overline{X}') such that $t' \in \Gamma(\overline{\mathcal{X}}', \mathcal{O}_{\overline{\mathcal{X}}'})$ is a lift of \bar{t}' and a morphism $\tilde{f} : ((X', \overline{X}', \overline{\mathcal{X}}'), t') \longrightarrow ((X, \overline{X}, \overline{\mathcal{X}}), t)$ with $\tilde{f}^*t = t'^n$ which is compatible with $f : (X', \overline{X}') \longrightarrow (X, \overline{X})$.*

Proof. In the situation of the lemma, we have a diagram

$$\begin{array}{ccc} (\overline{X}', Z') & \xrightarrow{f} & (\overline{X}, Z) \\ \downarrow & & \downarrow \\ (\mathrm{Spec} k[\bar{t}'], \{\bar{t}' = 0\}) & \xrightarrow{f_0} & (\mathrm{Spec} k[\bar{t}], \{\bar{t} = 0\}), \end{array}$$

where the vertical arrows are induced from \bar{t}, \bar{t}' and f_0 is induced by the ring homomorphism $k[\bar{t}] \longrightarrow k[\bar{t}'] : \bar{t} \mapsto \bar{t}'^n$. Then f is factorized as

$$(1.4) \quad (\overline{X}', Z') \longrightarrow (\overline{X}, Z) \times_{(\mathrm{Spec} k[\bar{t}], \{\bar{t}=0\})} (\mathrm{Spec} k[\bar{t}'], \{\bar{t}'=0\}) \longrightarrow (\overline{X}, Z).$$

Then f is log smooth by assumption and the second morphism in (1.4) is log etale because so is f_0 . Hence the first morphism in (1.4) is also log smooth by [16, 3.5]. Since it is strict by assumption, it induces the smooth morphism $g : \overline{X}' \longrightarrow \overline{X} \times_{\mathrm{Spec} k[\bar{t}]} \mathrm{Spec} k[\bar{t}']$ of affine schemes. Hence there exists a formal scheme $\overline{\mathcal{X}}'$ with $\overline{\mathcal{X}}' \otimes_{O_K} k = \overline{X}'$ and a smooth morphism $\tilde{g} : \overline{\mathcal{X}}' \longrightarrow \overline{\mathcal{X}} \times_{\mathrm{Spf} O_K\{\bar{t}\}, \tilde{f}_0} \mathrm{Spf} O_K\{\bar{t}'\}$ lifting g , where $\tilde{f}_0 : \mathrm{Spf} O_K\{\bar{t}'\} \longrightarrow \mathrm{Spf} O_K\{\bar{t}\}$ is the morphism induced by the ring

homomorphism $O_K\{t\} \longrightarrow O_K\{t'\}; t \mapsto t'^n$. Then $((X', \overline{X}', \overline{\mathcal{X}}'), t' := \tilde{g}^*t')$ defines a charted smooth standard small frame, and the composite

$$\overline{\mathcal{X}}' \xrightarrow{\tilde{g}} \overline{\mathcal{X}} \times_{\mathrm{Spf} O_K\{t\}, \tilde{f}_0} \mathrm{Spf} O_K\{t'\} \xrightarrow{\mathrm{proj.}} \overline{\mathcal{X}}$$

induces the morphism \tilde{f} of charted smooth standard small frames as in the statement of the lemma. So we are done. \square

Remark 1.13. In this remark, let the notation be as in Lemma 1.12 and we denote the zero locus of t (t') in $\overline{\mathcal{X}}$ ($\overline{\mathcal{X}}'$) by \mathcal{Z} (\mathcal{Z}'). Then, by looking the proof of Lemma 1.12 carefully, we see the following: The morphism $(\overline{\mathcal{X}}', \mathcal{Z}') \longrightarrow (\overline{\mathcal{X}}, \mathcal{Z})$ defining \tilde{f} is log smooth, and it is strict smooth when $n = 1$. Also, if the morphism $Z' \longrightarrow Z$ induced by f is an isomorphism, the morphism $\mathcal{Z}' \longrightarrow \mathcal{Z}$ induced by \tilde{f} is also an isomorphism because it is a lift of the morphism $Z' \longrightarrow Z$ to a morphism of p -adic smooth formal schemes over $\mathrm{Spf} O_K$. So, if the morphism $Z' \longrightarrow Z$ induced by f is an isomorphism and if we are given a charted smooth standard small frame with generic point $((X, \overline{X}, \overline{\mathcal{X}}), t, L)$, the morphism \tilde{f} can be enriched to a morphism of charted smooth standard small frames with generic points of the form $(\tilde{f}, \mathrm{id}) : ((X', \overline{X}', \overline{\mathcal{X}}'), t', L) \longrightarrow ((X, \overline{X}, \overline{\mathcal{X}}), t, L)$.

Let $j : X \hookrightarrow \overline{X}$ be an open immersion of smooth k -varieties such that $\overline{X} \setminus X =: Z$ is a simple normal crossing divisor. Then we have the log scheme (\overline{X}, Z) (see Convention for this notation) and the category of locally free log convergent isocrystals $\mathrm{Isoc}^{\mathrm{log}}(\overline{X}, Z)$ on (\overline{X}, Z) over K . We recall the notion of ‘having exponents in Σ ’ for an object in $\mathrm{Isoc}^{\mathrm{log}}(\overline{X}, Z)$, following [44, 3.7]. (We also introduce the notion of ‘having exponents in Σ with semisimple residues’.)

Definition 1.14. Let $X \hookrightarrow \overline{X}$ be an open immersion of smooth k -varieties such that $Z := \overline{X} \setminus X$ is a simple normal crossing divisor and let $Z = \bigcup_{i=1}^r Z_i$ be the decomposition of Z by irreducible components. Let $\Sigma = \prod_{i=1}^r \Sigma_i$ be a subset of \mathbb{Z}_p^r . Then we say that an object \mathcal{E} in $\mathrm{Isoc}^{\mathrm{log}}(\overline{X}, Z)$ has exponents in Σ (resp. has exponents in Σ with semisimple residues) if there exist an affine open covering $\overline{X} = \bigcup_{\alpha \in \Delta} \overline{U}_\alpha$ and charted standard small frames $((U_\alpha, \overline{U}_\alpha, \overline{\mathcal{X}}_\alpha), (t_{\alpha,1}, \dots, t_{\alpha,r}))$ enclosing $(U_\alpha, \overline{U}_\alpha)$ ($\alpha \in \Delta$, where we put $U_\alpha := X \cap \overline{U}_\alpha$) such that, for any $\alpha \in \Delta$ and any i ($1 \leq i \leq r$), all the exponents of the log- ∇ -module $E_{\mathcal{E},\alpha}$ on $\overline{\mathcal{X}}_{\alpha,K}$ induced by \mathcal{E} along the locus $\{t_{\alpha,i} = 0\}$ are contained in Σ_i (resp. all the exponents of the log- ∇ -module $E_{\mathcal{E},\alpha}$ on $\overline{\mathcal{X}}_{\alpha,K}$ induced by \mathcal{E} along the locus $\{t_{\alpha,i} = 0\}$ are contained in Σ_i and there exists some polynomial $P_{\alpha,i}(x) \in \mathbb{Z}_p[x]$ without any multiple roots such that $P_{\alpha,i}(\mathrm{res}_i) = 0$ holds, where res_i is the residue of $E_{\mathcal{E},\alpha}$ along $\{t_{\alpha,i} = 0\}$). We denote the category of objects in $\mathrm{Isoc}^{\mathrm{log}}(\overline{X}, Z)$ having exponents in Σ (resp. having exponents in Σ with semisimple residues) by $\mathrm{Isoc}^{\mathrm{log}}(\overline{X}, Z)'_\Sigma$ (resp. $\mathrm{Isoc}^{\mathrm{log}}(\overline{X}, Z)'_{\Sigma\text{-ss}}$).

As a variant of [44, 3.8], we can prove the following:

Lemma 1.15. *Let $(X, \overline{X}), Z := \bigcup_{i=1}^r Z_i, \Sigma$ be as above and let \mathcal{E} be an object in the category $\text{Isoc}^{\log}(\overline{X}, Z)$. Then:*

- (1) *\mathcal{E} has exponents in Σ (resp. has exponents in Σ with semisimple residues) if and only if the following condition is satisfied: For any affine open subscheme $\overline{U} \hookrightarrow \overline{X}$, any charted standard small frame $((U, \overline{U}, \overline{\mathcal{X}}), (t_1, \dots, t_r))$ enclosing (U, \overline{U}) (where we put $U := X \cap \overline{U}$) and for any i ($1 \leq i \leq r$), all the exponents of the log- ∇ -module $E_{\mathcal{E}}$ on $\overline{\mathcal{X}}_K$ induced by \mathcal{E} along the locus $\{t_i = 0\}$ are contained in Σ_i (resp. all the exponents of the log- ∇ -module $E_{\mathcal{E}}$ on $\overline{\mathcal{X}}_K$ induced by \mathcal{E} along the locus $\{t_i = 0\}$ are contained in Σ_i and there exists some polynomial $P_i(x) \in \mathbb{Z}_p[x]$ without multiple roots such that $P_i(\text{res}_i) = 0$ holds, where res_i is the residue of $E_{\mathcal{E}}$ along $\{t_i = 0\}$).*
- (2) *\mathcal{E} has exponents in Σ (resp. has exponents in Σ with semisimple residues) if and only if the following condition is satisfied: there exist affine open subschemes $\overline{U}^{(\alpha)} \subseteq \overline{X} \setminus Z_{\text{sing}}$ (where Z_{sing} is the set of singular points of Z) containing the generic point of Z_{α} ($1 \leq \alpha \leq r$), charted smooth standard small frames $((U^{(\alpha)}, \overline{U}^{(\alpha)}, \overline{\mathcal{X}}^{(\alpha)}), t^{(\alpha)})$ enclosing $(U^{(\alpha)}, \overline{U}^{(\alpha)})$ (where we put $U^{(\alpha)} := X \cap \overline{U}^{(\alpha)}$), such that, for any α ($1 \leq \alpha \leq r$), all the exponents of the log- ∇ -module $E_{\mathcal{E}}^{(\alpha)}$ on $\overline{\mathcal{X}}_K^{(\alpha)}$ induced by \mathcal{E} along the locus $\{t^{(\alpha)} = 0\}$ are contained in Σ_{α} (resp. all the exponents of the log- ∇ -module $E_{\mathcal{E}}^{(\alpha)}$ on $\overline{\mathcal{X}}_K^{(\alpha)}$ induced by \mathcal{E} along the locus $\{t^{(\alpha)} = 0\}$ are contained in Σ_{α} and there exists some polynomial $P^{(\alpha)}(x) \in \mathbb{Z}_p[x]$ without multiple roots such that $P^{(\alpha)}(\text{res}^{(\alpha)}) = 0$ holds, where $\text{res}^{(\alpha)}$ is the residue of $E_{\mathcal{E}, \alpha}$ along $\{t^{(\alpha)} = 0\}$).*

Proof. First we prove (1). In the case of ‘having exponents in Σ ’, this is proven in [44, 3.8]. In the case of ‘having exponents in Σ with semisimple residues’, we can prove the lemma in the same way, as follows: Let \mathcal{E} be an object in $\text{Isoc}^{\log}(\overline{X}, Z)$ with exponents in Σ with semisimple residues and take $((U, \overline{U}, \overline{\mathcal{X}}), (t_1, \dots, t_r))$, $E_{\mathcal{E}}$ as in (1). It suffices to prove that there exists some polynomial $P_i(x) \in \mathbb{Z}_p[x]$ without multiple roots such that $P_i(\text{res}_i) = 0$ holds, where res_i is the residue of $E_{\mathcal{E}}$ along $\{t_i = 0\}$.

Let $\overline{X} = \bigcup_{\alpha \in \Delta} \overline{U}_{\alpha}$ and take $((U_{\alpha}, \overline{U}_{\alpha}, \overline{\mathcal{X}}_{\alpha}), (t_{\alpha,1}, \dots, t_{\alpha,r}))$, $E_{\mathcal{E}, \alpha}$ and $P_{\alpha,i}(x) \in \mathbb{Z}_p[x]$ as in Definition 1.14. By shrinking $\overline{\mathcal{X}}$ and $\overline{\mathcal{X}}_{\alpha}$ appropriately, we may assume that $\overline{U} = \overline{U}_{\alpha}$ for some α . Then, in the proof of [44, 3.8], we constructed the diagram

$$(1.5) \quad \overline{\mathcal{X}}_{\alpha, K} =]\overline{U}_{[\overline{\mathcal{X}}_{\alpha} \xleftarrow{\pi_1}]\overline{U}_{[\overline{\mathcal{X}}_{\alpha} \times \overline{\mathcal{X}}}^{\log} \cong]\overline{U}_{[\hat{\mathbb{A}}^d \times \overline{\mathcal{X}}} \xleftarrow{s}]\overline{U}_{[\overline{\mathcal{X}} = \overline{\mathcal{X}}_K$$

and proved that the residue of $E_{\mathcal{E}, \alpha}$ along $\{t_{\alpha,i} = 0\}$ is pulled back by (1.5) to the residue res_i of $E_{\mathcal{E}}$ along $\{t_i = 0\}$. So we have $P_{\alpha,i}(\text{res}_i) = 0$ and so we have proved (1).

Next we prove (2). Let us take $((U, \overline{U}, \overline{\mathcal{X}}), (t_1, \dots, t_r))$, $E_{\mathcal{E}}$ as in (1). Then it suffices to prove that all the exponents of $E_{\mathcal{E}}$ along $\{t_i = 0\}$ are contained in Σ_i (and there exists some polynomial $P_i(x) \in \mathbb{Z}_p[x]$ without multiple roots such that

$P_i(\text{res}_i) = 0$ holds, where res_i is the residue of $E_{\mathcal{E}}$ along $\{t_i = 0\}$). Let us consider the intersection $\overline{U} \cap \overline{U}^{(i)} \cap Z_i$. When it is empty, we have $\overline{U} \cap Z_i = \emptyset$ and so the locus $\{t_i = 0\}$ is empty. Hence the assertion we have to prove is vacuous in this case. So let us consider the case where $\overline{U} \cap \overline{U}^{(i)} \cap Z_i$ is non-empty. Then we can take a non-empty affine open formal subscheme $\overline{\mathcal{X}}_i$ of $\overline{\mathcal{X}}$ with $\overline{\mathcal{X}}_i \otimes_{O_K} k \subseteq \overline{U} \cap \overline{U}^{(i)}$ and $\overline{\mathcal{X}}_i \cap \{t_i = 0\}$ non-empty. Let $F_{\mathcal{E}} := \mathcal{E}\text{nd}(E_{\mathcal{E}})$. Then, by the argument in [44, between 1.2 and 1.3], the restriction map

$$\Gamma(\{t_i = 0\}, F_{\mathcal{E}}) \longrightarrow \Gamma(\overline{\mathcal{X}}_{i,K} \cap \{t_i = 0\}, F_{\mathcal{E}})$$

is injective. So, it suffices to show that $E_{\mathcal{E}}|_{\overline{\mathcal{X}}_{i,K}}$ has exponents in Σ_i (and $P^{(i)}(\text{res}_i) = 0$), that is, we may assume that $\overline{U} \subseteq \overline{U}^{(i)}$ to prove the desired assertion. Then, by shrinking $\overline{U}^{(i)}$, we may assume that $\overline{U} = \overline{U}^{(i)}$. In this case, we have the diagram (1.5) with $\overline{\mathcal{X}}_{\alpha,K}$ replaced by $\overline{\mathcal{X}}_K^{(i)}$. So the residue of $E_{\mathcal{E}}^{(i)}$ along $\{t^{(i)} = 0\}$ is pulled back to the residue res_i of $E_{\mathcal{E}}$ along $\{t_i = 0\}$. Hence $E_{\mathcal{E}}$ has exponents in Σ_i (and $P^{(i)}(\text{res}_i) = 0$) and thus we have also proved (2). \square

Next we recall the notion of ‘having Σ -unipotent generic monodromy’ for an object in $\text{Isoc}^{\dagger}(X, \overline{X})$. We introduce also the notion of ‘having Σ -semisimple generic monodromy’.

Definition 1.16. *Let $X \hookrightarrow \overline{X}$ be an open immersion of smooth k -varieties such that $Z := \overline{X} \setminus X$ is a simple normal crossing divisor. Let $Z = \bigcup_{i=1}^r Z_i$ be the decomposition of Z by irreducible components and let Z_{sing} be the set of singular points of Z . Let $\Sigma = \prod_{i=1}^r \Sigma_i$ be a subset of \mathbb{Z}_p^r . Then we say that an overconvergent isocrystal \mathcal{E} on (X, \overline{X}) over K has Σ -unipotent generic monodromy (resp. Σ -semisimple generic monodromy) if there exist affine open subschemes $\overline{U}_{\alpha} \subseteq \overline{X} \setminus Z_{\text{sing}}$ containing the generic point of Z_{α} ($1 \leq \alpha \leq r$), charted smooth standard small frames with generic point $((U_{\alpha}, \overline{U}_{\alpha}, \overline{\mathcal{X}}_{\alpha}), t_{\alpha}, L_{\alpha})$ enclosing $(U_{\alpha}, \overline{U}_{\alpha})$ (where we put $U_{\alpha} := X \cap \overline{U}_{\alpha}$) satisfying the following condition: For any $1 \leq \alpha \leq r$, there exists some $\lambda \in (0, 1) \cap \Gamma^*$ such that the ∇ -module $E_{\mathcal{E}, \alpha}$ associated to \mathcal{E} is defined on $\{x \in \overline{\mathcal{X}}_{\alpha,K} \mid |t_{\alpha}(x)| \geq \lambda\}$ and that the restriction of $E_{\mathcal{E}, \alpha}$ to $A_{L_{\alpha}}^1[\lambda, 1)$ is Σ_{α} -unipotent (resp. Σ_{α} -semisimple).*

We denote the category of objects in $\text{Isoc}^{\dagger}(X, \overline{X})$ having Σ -unipotent generic monodromy (resp. having Σ -semisimple generic monodromy) by $\text{Isoc}^{\dagger}(X, \overline{X})'_{\Sigma}$ (resp. $\text{Isoc}^{\dagger}(X, \overline{X})'_{\Sigma\text{-ss}}$).

Note that the notion of having Σ -unipotent generic monodromy (Σ -semisimple generic monodromy) depends only on the image $\overline{\Sigma}$ of Σ in $\mathbb{Z}_p^r/\mathbb{Z}^r$ in the sense that \mathcal{E} has Σ -unipotent generic monodromy (Σ -semisimple generic monodromy) if and only if \mathcal{E} has $\tau(\overline{\Sigma})$ -unipotent generic monodromy ($\tau(\overline{\Sigma})$ -semisimple generic monodromy) for some (or any) section $\tau : \mathbb{Z}_p^r/\mathbb{Z}^r \longrightarrow \mathbb{Z}_p$ of the form $\tau = \prod_{i=1}^r \tau_i$ of the canonical projection $\mathbb{Z}_p^r \longrightarrow \mathbb{Z}_p^r/\mathbb{Z}^r$. (See [44, 1.4] or the paragraph after Definition 1.2.) Hence it is allowed to say that \mathcal{E} has $\overline{\Sigma}$ -unipotent generic monodromy ($\overline{\Sigma}$ -semisimple

generic monodromy) and denote by $\text{Isoc}^\dagger(X, \overline{X})'_{\overline{\Sigma}}$ (resp. $\text{Isoc}^\dagger(X, \overline{X})'_{\overline{\Sigma}\text{-ss}}$) by abuse of terminology.

Then the main theorem of [44] is described as follows. (We add also the ‘semisimple’ variant of the theorem.)

Theorem 1.17. *Let $X \hookrightarrow \overline{X}$ be an open immersion of smooth k -varieties such that $Z := \overline{X} \setminus X$ is a simple normal crossing divisor and let $Z = \bigcup_{i=1}^r Z_i$ be the decomposition of Z into irreducible components. Let $\overline{\Sigma} := \prod_{i=1}^r \overline{\Sigma}_i$ be a subset of $\mathbb{Z}_p^r/\mathbb{Z}^r$ which is (NLD) and let $\tau : \mathbb{Z}_p^r/\mathbb{Z}^r \rightarrow \mathbb{Z}_p$ be a section of the form $\tau = \prod_{i=1}^r \tau_i$ of the canonical projection $\mathbb{Z}_p^r \rightarrow \mathbb{Z}_p^r/\mathbb{Z}^r$. Then we have the canonical equivalences of categories*

$$(1.6) \quad j^\dagger : \text{Isoc}^{\log}(\overline{X}, Z)'_{\tau(\overline{\Sigma})} \xrightarrow{=} \text{Isoc}^\dagger(X, \overline{X})'_{\overline{\Sigma}},$$

$$(1.7) \quad j^\dagger : \text{Isoc}^{\log}(\overline{X}, Z)'_{\tau(\overline{\Sigma})\text{-ss}} \xrightarrow{=} \text{Isoc}^\dagger(X, \overline{X})'_{\overline{\Sigma}\text{-ss}},$$

which are defined by the restriction.

Proof. The equivalence (1.6) follows from [44, 3.12, 3.16, 3.17]. Let us prove the equivalence (1.7). First let us take $\mathcal{E} \in \text{Isoc}^{\log}(\overline{X}, Z)'_{\tau(\overline{\Sigma})\text{-ss}}$ and let us take open subschemes $\overline{U}_\alpha \subseteq \overline{X} \setminus Z_{\text{sing}}$ containing the generic point of Z_α ($1 \leq \alpha \leq r$) and charted smooth standard small frames with generic point $((U_\alpha, \overline{U}_\alpha, \overline{\mathcal{X}}_\alpha), t_\alpha, L_\alpha)$ enclosing $(U_\alpha, \overline{U}_\alpha)$ (where we put $U_\alpha := X \cap \overline{U}_\alpha$). Let $E_{\mathcal{E}, \alpha}$ be the log- ∇ -module on $\overline{\mathcal{X}}_{\alpha, K}$ induced by \mathcal{E} , let $E_{\mathcal{E}, L, \alpha}$ be the restriction of $E_{\mathcal{E}, \alpha}$ to $A_{L_\alpha}^1[0, 1)$ and let \mathcal{Z}_α be the zero locus of t_α in $\overline{\mathcal{X}}_\alpha$. Then, by [44, 3.6, 2.12], $E_{\mathcal{E}, \alpha}|_{\mathcal{Z}_{\alpha, K} \times A_K^1[0, 1)}$ is Σ_α -unipotent. So $E_{\mathcal{E}, L, \alpha}$ is also Σ_α -unipotent and hence $E_{\mathcal{E}, L, \alpha} = \mathcal{U}_{[0, 1)}(E, \partial)$ for some $(E, \partial) \in \text{ULNM}_{A_{L_\alpha}^1[0, 0]}$. On the other hand, by Lemma 1.15, there exists some $P_\alpha(x) \in \mathbb{Z}_p[x]$ without multiple roots such that $P_\alpha(\text{res}_\alpha) = 0$, where res_α denotes the residue of $E_{\mathcal{E}, \alpha}$ along $\mathcal{Z}_{\alpha, K}$. Then the residue $\text{res}_{L, \alpha}$ of $E_{\mathcal{E}, L, \alpha}$ satisfies the same equation. Note that, when we restrict $E_{\mathcal{E}, L, \alpha}$ to the locus $\{t = 0\}$ (where t denotes the coordinate of $A_{L_\alpha}^1[0, 1)$), we obtain (E, ∂) . So we have the equation $P_\alpha(\partial) = 0$, that is, ∂ acts semisimply on E . Hence $E_{\mathcal{E}, L, \alpha} = \mathcal{U}_{[0, 1)}(E, \partial)$ is Σ -semisimple and so is the restriction of $E_{\mathcal{E}, L, \alpha}$ to $A_{L_\alpha}^1[\lambda, 1)$ for any $\lambda \in [0, 1) \cap \Gamma^*$. Therefore, we have $j^\dagger \mathcal{E} \in \text{Isoc}^\dagger(X, \overline{X})'_{\overline{\Sigma}\text{-ss}}$, that is, the functor (1.7) is well-defined.

The full faithfulness of (1.7) follows from that of (1.6). Let us prove the essential surjectivity of (1.7). Let us take $\mathcal{E}' \in \text{Isoc}^\dagger(X, \overline{X})'_{\overline{\Sigma}\text{-ss}}$ and take $\mathcal{E} \in \text{Isoc}^{\log}(\overline{X}, Z)'_{\tau(\overline{\Sigma})}$ with $j^\dagger \mathcal{E} = \mathcal{E}'$. Let us take $((U_\alpha, \overline{U}_\alpha, \overline{\mathcal{X}}_\alpha), t_\alpha, L_\alpha)$, \mathcal{Z}_α and λ such that the ∇ -module $E_{\mathcal{E}', \alpha}$ associated to \mathcal{E}' is defined on $\{x \in \overline{\mathcal{X}}_{\alpha, K} \mid |t_\alpha(x)| \geq \lambda\}$ and that the restriction $E_{\mathcal{E}', L, \alpha}$ of $E_{\mathcal{E}', \alpha}$ to $A_{L_\alpha}^1[\lambda, 1)$ is Σ_α -semisimple. By definition of \mathcal{E} , $E_{\mathcal{E}', L, \alpha}$ extends to the log- ∇ -module $E_{\mathcal{E}, L, \alpha}$ on $A_{L_\alpha}^1[0, 1)$ induced by \mathcal{E} and it is Σ_α -unipotent by [44, 3.6, 2.12]. Now let us note the equivalences

$$\begin{aligned} \text{ULNM}_{A_{L_\alpha}[0, 0], \Sigma_\alpha} &\xrightarrow{\mathcal{U}_{[0, 1)}} \text{ULNM}_{A_{L_\alpha}[0, 1), \Sigma_\alpha} \xrightarrow{=} \text{ULNM}_{A_{L_\alpha}(\lambda, 1), \Sigma_\alpha}, \\ \text{SLNM}_{A_{L_\alpha}[0, 0], \Sigma_\alpha} &\xrightarrow{\mathcal{U}_{[0, 1)}} \text{SLNM}_{A_{L_\alpha}[0, 1), \Sigma_\alpha} \xrightarrow{=} \text{SLNM}_{A_{L_\alpha}(\lambda, 1), \Sigma_\alpha} \end{aligned}$$

induced by (1.5): By the first equivalences, we see that there exists an object $(E, \partial) \in \text{ULNM}_{A_{L_\alpha}[0,0], \Sigma_\alpha}$ which is sent to $E_{\mathcal{E}, L, \alpha} \in \text{ULNM}_{A_{L_\alpha}[0,1], \Sigma_\alpha}$ and it is sent to $E_{\mathcal{E}', L, \alpha}|_{A_{L_\alpha}(\lambda, 1)}$. Then, since $E_{\mathcal{E}', L, \alpha}|_{A_{L_\alpha}(\lambda, 1)}$ belongs to $\text{SLNM}_{A_{L_\alpha}(\lambda, 1), \Sigma_\alpha}$, we see by the second equivalences that (E, ∂) is actually in $\text{SLNM}_{A_{L_\alpha}[0,0], \Sigma_\alpha}$. Hence we have $E_{\mathcal{E}, L, \alpha} \in \text{SLNM}_{A_{L_\alpha}[0,1], \Sigma_\alpha}$ and this implies that there exists a polynomial $P_\alpha(x) \in \mathbb{Z}_p[x]$ without multiple roots such that the residue res of $E_{\mathcal{E}, L, \alpha}$ along $\{t = 0\}$ (where t denotes the coordinate of $A_{L_\alpha}^1[0, 1]$) satisfies $P_\alpha(\text{res}) = 0$. Then, since the restriction map $\Gamma(\mathcal{Z}_{\alpha, K}, \mathcal{E}\text{nd}(E_{\mathcal{E}, \alpha}|_{\mathcal{Z}_{\alpha, K}})) \longrightarrow \Gamma(\text{Spm } L_\alpha, \mathcal{E}\text{nd}(E_{\mathcal{E}, L, \alpha}|_{\{t=0\}}))$ is injective, the residue res_α of $E_{\mathcal{E}, \alpha}$ along $\mathcal{Z}_{\alpha, K}$ also satisfies $P_\alpha(\text{res}_\alpha) = 0$. Hence \mathcal{E} belongs to $\text{Isoc}^{\log}(\overline{X}, Z)'_{\tau(\overline{\Sigma})\text{-ss}}$, by Lemma 1.15. So we are done. \square

We have also the following variant of the full faithfulness of the functor (1.7), which is useful in this paper:

Proposition 1.18. *Let $X \hookrightarrow \overline{X}$, $Z := \overline{X} \setminus X$ be as above and let $\overline{\Sigma} := \prod_{i=1}^r \overline{\Sigma}_i$ be a subset of $\mathbb{Z}_p^r/\mathbb{Z}^r$ which is (NLD). Let $\tau_i : \mathbb{Z}_p^r/\mathbb{Z}^r \longrightarrow \mathbb{Z}_p^r$ ($i = 1, 2$) be sections of the form $\tau_i = \prod_{j=1}^r \tau_{ij}$ of the canonical projection $\mathbb{Z}_p^r \longrightarrow \mathbb{Z}_p^r/\mathbb{Z}^r$ such that for any j and any $\xi \in \overline{\Sigma}_j$, $\tau_{1j}(\xi) \geq \tau_{2j}(\xi)$. Then, for any $\mathcal{E}_i \in \text{Isoc}^{\log}(\overline{X}, Z)'_{\tau_i(\overline{\Sigma})}$ ($i = 1, 2$), the homomorphism*

$$\text{Hom}(\mathcal{E}_1, \mathcal{E}_2) \longrightarrow \text{Hom}(j^\dagger \mathcal{E}_1, j^\dagger \mathcal{E}_2)$$

is an isomorphism.

Proof. Since we may work Zariski locally on \overline{X} , we may assume that there exists a charted standard small frame $((X, \overline{X}, \overline{\mathcal{X}}), (t_1, \dots, t_r))$ enclosing (X, \overline{X}) . Let \tilde{E}_i (resp. E_i) be the log- ∇ -module (resp. ∇ -module) on $\overline{\mathcal{X}}_K$ (resp. on a strict neighborhood of $]X[_{\overline{\mathcal{X}}}$ in $\overline{\mathcal{X}}_K$) induced by \mathcal{E}_i (resp. $j^\dagger \mathcal{E}_i$) ($i = 1, 2$). We may assume that E_i 's are both defined on $\mathfrak{Y} := \{x \in \overline{\mathcal{X}}_K \mid \forall i, |t_i(x)| \geq \lambda\}$. It suffices to prove the isomorphism

$$(1.8) \quad \text{Hom}_{\overline{\mathcal{X}}_K}(\tilde{E}_1, \tilde{E}_2) \xrightarrow{=} \text{Hom}_{\mathfrak{Y}}(E_1, E_2).$$

(Here Hom denotes the set of homomorphism as (log-) ∇ -modules.) Let us consider the admissible covering $\overline{\mathcal{X}}_K = \bigcup_{I \subseteq \{1, \dots, r\}} \mathfrak{X}_I$, where \mathfrak{X}_I is defined by

$$\mathfrak{X}_I := \{x \in \overline{\mathcal{X}}_K \mid |t_i(x)| < 1 \ (i \in I), |t_i(x)| \geq \lambda \ (i \notin I)\}.$$

This covering induces the admissible covering $\mathfrak{Y} = \bigcup_{I \subseteq \{1, \dots, r\}} \mathfrak{Y}_I$, where

$$\mathfrak{Y}_I := \{x \in \overline{\mathcal{X}}_K \mid \lambda \leq |t_i(x)| < 1 \ (i \in I), |t_i(x)| \geq \lambda \ (i \notin I)\}.$$

For $i = 1, 2$, \tilde{E}_i is $\prod_{j \in I} \tau_i(\Sigma_{ij})$ -unipotent on

$$\mathfrak{X}_I = \{x \in \overline{\mathcal{X}}_K \mid t_i(x) = 0 \ (i \in I), |t_i(x)| \geq \lambda \ (i \notin I)\} \times A^{|I|}[0, 1]$$

by [44, 3.6, 2.12]. Therefore, we have the isomorphism

$$\text{Hom}_{\mathfrak{X}_I}(\tilde{E}_1, \tilde{E}_2) \xrightarrow{=} \text{Hom}_{\mathfrak{Y}_I}(E_1, E_2)$$

by Proposition 1.6. By noting the equalities

$$\begin{aligned}\mathfrak{X}_I \cap \mathfrak{X}_J &= \{x \in P_K \mid \lambda \leq |t_i(x)| < 1 \ (i \in (I \cup J) \setminus (I \cap J)), \\ &\quad \lambda \leq |t_i(x)| \ (i \notin I \cup J)\} \times A_K^{|I \cap J|}[0, 1), \\ \mathfrak{Y}_I \cap \mathfrak{Y}_J &= \{x \in P_K \mid \lambda \leq |t_i(x)| < 1 \ (i \in (I \cup J) \setminus (I \cap J)), \\ &\quad \lambda \leq |t_i(x)| \ (i \notin I \cup J)\} \times A_K^{|I \cap J|}[\lambda, 1),\end{aligned}$$

we also see the isomorphism

$$\mathrm{Hom}_{\mathfrak{X}_I \cap \mathfrak{X}_J}(\tilde{E}_1, \tilde{E}_2) \xrightarrow{\cong} \mathrm{Hom}_{\mathfrak{Y}_I \cap \mathfrak{Y}_J}(E_1, E_2)$$

by the same argument. So we have the isomorphism (1.8). \square

Finally in this section, we give a slightly different formulation concerning the definition of log convergent isocrystals having exponents in Σ (with semisimple residues) and overconvergent isocrystals having Σ -unipotent (Σ -semisimple) generic monodromy. The formulation given below is useful when we discuss the functoriality.

Definition 1.19. *Let $X \hookrightarrow \overline{X}$ be an open immersion of smooth k -varieties such that $Z = \overline{X} \setminus X$ is a simple normal crossing divisor and assume that we are given a family of simple normal crossing subdivisors $\{Z_i\}_{i=1}^r$ of Z with $Z = \sum_{i=1}^r Z_i$ (we call such a family $\{Z_i\}_{i=1}^r$ a decomposition of Z) and a subset $\Sigma = \prod_{i=1}^r \Sigma_i$ of \mathbb{Z}_p^r . Let $Z_i = \bigcup_{j=1}^{r_i} Z_{ij}$ be the decomposition of Z_i into irreducible components and let us put $\Sigma' := \prod_{i=1}^r \Sigma_i^{r_i} \subseteq \mathbb{Z}_p^{\sum_{i=1}^r r_i}$. Then we say that an object \mathcal{E} in $\mathrm{Isoc}^{\log}(\overline{X}, Z)$ has exponents in Σ (resp. has exponents in Σ with semisimple residues) with respect to the decomposition $\{Z_i\}_{i=1}^r$ when it has exponents in Σ' (resp. it has exponents in Σ' with semisimple residues) in the sense of Definition 1.14. We denote the category of objects in $\mathrm{Isoc}^{\log}(\overline{X}, Z)$ having exponents in Σ (resp. having exponents in Σ with semisimple residues) with respect to the decomposition $\{Z_i\}_{i=1}^r$ by $\mathrm{Isoc}^{\log}(\overline{X}, Z)_{\Sigma}$ (resp. $\mathrm{Isoc}^{\log}(\overline{X}, Z)_{\Sigma\text{-ss}}$).*

Definition 1.20. *Let $X \hookrightarrow \overline{X}$ be an open immersion of smooth k -varieties such that $Z = \overline{X} \setminus X$ is a simple normal crossing divisor and assume that we are given a decomposition $\{Z_i\}_{i=1}^r$ of Z in the sense of Definition 1.19 and a subset $\Sigma = \prod_{i=1}^r \Sigma_i$ of \mathbb{Z}_p^r or $\mathbb{Z}_p^r/\mathbb{Z}^r$. Let $Z_i = \bigcup_{j=1}^{r_i} Z_{ij}$ be the decomposition of Z_i into irreducible components and let us put $\Sigma' := \prod_{i=1}^r \Sigma_i^{r_i}$, which is a subset of $\mathbb{Z}_p^{\sum_{i=1}^r r_i}$ or $(\mathbb{Z}_p/\mathbb{Z})^{\sum_{i=1}^r r_i}$. Then we say that an object \mathcal{E} in $\mathrm{Isoc}^{\dagger}(X, \overline{X})$ has Σ -unipotent generic monodromy (resp. has Σ -semisimple generic monodromy) with respect to the decomposition $\{Z_i\}_{i=1}^r$ when it has Σ' -unipotent generic monodromy (resp. it has Σ' -semisimple generic monodromy) in the sense of Definition 1.16. We denote the category of objects in $\mathrm{Isoc}^{\dagger}(X, \overline{X})$ having Σ -unipotent generic monodromy (resp. Σ -semisimple generic monodromy) with respect to the decomposition $\{Z_i\}_{i=1}^r$ by $\mathrm{Isoc}^{\dagger}(X, \overline{X})_{\Sigma}$ (resp. $\mathrm{Isoc}^{\dagger}(X, \overline{X})_{\Sigma\text{-ss}}$).*

Note that, when each Z_i is irreducible and non-empty, we have $\text{Isoc}^{\log}(\overline{X}, Z)_{\Sigma} = \text{Isoc}^{\log}(\overline{X}, Z)_{\Sigma}'$, $\text{Isoc}^{\log}(\overline{X}, Z)_{\Sigma(-ss)} = \text{Isoc}^{\log}(\overline{X}, Z)_{\Sigma(-ss)}'$, $\text{Isoc}^{\dagger}(X, \overline{X})_{\Sigma} = \text{Isoc}^{\dagger}(X, \overline{X})_{\Sigma}'$ and $\text{Isoc}^{\dagger}(X, \overline{X})_{\Sigma(-ss)} = \text{Isoc}^{\dagger}(X, \overline{X})_{\Sigma(-ss)}'$. We prove here certain functoriality results for the categories of the form $\text{Isoc}^{\log}(\overline{X}, Z)_{\Sigma(-ss)}$.

Proposition 1.21. *Let X, \overline{X}, Z be as above and assume we are given a decomposition $\{Z_i\}_{i=1}^r$ of Z and a subset $\Sigma = \prod_{i=1}^r \Sigma_i$ of \mathbb{Z}_p^r . Let $f : \overline{X}' \rightarrow \overline{X}$ one of the following:*

- (1) *f is an open immersion such that the image of \overline{X}' contains all the generic points of Z .*
- (2) *f is the morphism of the form $\overline{X}' = \coprod_{\beta} \overline{X}_{\beta} \rightarrow \overline{X}$ for an open covering $\overline{X} = \bigcup_{\beta} \overline{X}_{\beta}$ by finite number of open subschemes.*

Let us put $Z' := Z \times_{\overline{X}} \overline{X}'$ and consider that we are given the decomposition $\{Z'_i\}_{i=1}^r = \{Z_i \times_{\overline{X}} \overline{X}'\}$ of Z' . Then, for an object \mathcal{E} in $\text{Isoc}^{\log}(\overline{X}, Z)$, it is actually contained in $\text{Isoc}^{\log}(\overline{X}, Z)_{\Sigma(-ss)}$ if and only if $f^*\mathcal{E} \in \text{Isoc}^{\log}(\overline{X}', Z')$ is contained in $\text{Isoc}^{\log}(\overline{X}', Z')_{\Sigma(-ss)}$.

Proof. We may assume that each Z_i is irreducible (and so $Z = \bigcup_{i=1}^r Z_i$ is the decomposition of Z into irreducible components). First consider the case (1). In this case, $Z' = \bigcup_{i=1}^r Z'_i$ is also the decomposition of Z into irreducible components. If we assume that \mathcal{E} is in the category $\text{Isoc}^{\log}(\overline{X}, Z)_{\Sigma(-ss)}$, we can take charted smooth standard small frames $((U^{(\alpha)}, \overline{U}^{(\alpha)}, \overline{\mathcal{X}}^{(\alpha)}), t^{(\alpha)})$ ($1 \leq \alpha \leq r$) satisfying the conclusion of Lemma 1.15(2). Then, by shrinking $\overline{\mathcal{X}}^{(\alpha)}$, we can assume that each $\overline{U}^{(\alpha)}$ is contained in \overline{X}' . Then we have $f^*\mathcal{E} \in \text{Isoc}^{\log}(\overline{X}', Z')_{\Sigma(-ss)}$ by Lemma 1.15(2). Conversely, if we assume that $f^*\mathcal{E}$ is in the category $\text{Isoc}^{\log}(\overline{X}', Z')_{\Sigma(-ss)}$, we can take charted smooth standard small frames $((U^{(\alpha)}, \overline{U}^{(\alpha)}, \overline{\mathcal{X}}^{(\alpha)}), t^{(\alpha)})$ ($1 \leq \alpha \leq r$) for (\overline{X}', Z') and $f^*\mathcal{E}$ satisfying the conclusion of Lemma 1.15(2). Then these charted smooth standard small frames satisfy the conclusion of Lemma 1.15(2) for (\overline{X}, Z) and \mathcal{E} . So \mathcal{E} is in the category $\text{Isoc}^{\log}(\overline{X}, Z)_{\Sigma(-ss)}$. So we are done in this case.

Next consider the case (2). In this case, $Z'_i = \bigcup_{\substack{\beta \\ Z'_i \cap \overline{X}_{\beta} \neq \emptyset}} Z'_i \cap \overline{X}_{\beta}$ gives the decomposition of Z'_i into irreducible components. If we assume that \mathcal{E} is in the category $\text{Isoc}^{\log}(\overline{X}, Z)_{\Sigma(-ss)}$, we can take charted smooth standard small frames $((U^{(\alpha)}, \overline{U}^{(\alpha)}, \overline{\mathcal{X}}^{(\alpha)}), t^{(\alpha)})$ ($1 \leq \alpha \leq r$) satisfying the conclusion of Lemma 1.15(2). Then

$$((U^{(\alpha)} \cap \overline{X}_{\beta}, \overline{U}^{(\alpha)} \cap \overline{X}_{\beta}, \overline{\mathcal{X}}_{\beta}^{(\alpha)}), t^{(\alpha)})$$

(where $\overline{\mathcal{X}}_{\beta}^{(\alpha)}$ is the open formal subscheme of $\overline{\mathcal{X}}^{(\alpha)}$ whose special fiber is equal to $\overline{U}^{(\alpha)} \cap \overline{X}_{\beta}$) for (α, β) with $Z'_i \cap \overline{X}_{\beta} \neq \emptyset$ satisfies the conclusion of Lemma 1.15(2) for (\overline{X}', Z') , $f^*\mathcal{E}$ and $\Sigma' := \prod_{\substack{(\alpha, \beta) \\ Z'_i \cap \overline{X}_{\beta} \neq \emptyset}} \Sigma_{\alpha}$. So we have $f^*\mathcal{E} \in \text{Isoc}^{\log}(\overline{X}', Z')_{\Sigma(-ss)}$.

Conversely, if $f^*\mathcal{E}$ is in $\text{Isoc}^{\log}(\overline{X}', Z')_{\Sigma(-\text{ss})}$, we can take charted smooth standard small frames $((U^{(\alpha,\beta)}, \overline{U}^{(\alpha,\beta)}, \overline{\mathcal{X}}^{(\alpha,\beta)}), t^{(\alpha,\beta)})$ ((α, β) runs through the indices with $Z'_\alpha \cap \overline{X}_\beta \neq \emptyset$) with $U^{(\alpha,\beta)}$ containing the generic point of $Z'_\alpha \cap \overline{X}_\beta$ such that they satisfy the conclusion of Lemma 1.15(2) for $(\overline{X}', Z'), f^*\mathcal{E}$ and Σ' . Noting the fact that $Z'_\alpha \cap \overline{X}_\beta$ is nonempty for some β for any $1 \leq \alpha \leq r$, we see that these charted smooth standard small frames satisfy the conclusion of Lemma 1.15(2) for (\overline{X}, Z) and \mathcal{E} . So \mathcal{E} is in the category $\text{Isoc}^{\log}(\overline{X}, Z)_{\Sigma(-\text{ss})}$. So we are done. \square

Proposition 1.22. *Let $X \hookrightarrow \overline{X}$ ($X' \hookrightarrow \overline{X}'$) be an open immersion of smooth k -varieties such that $Z = \overline{X} \setminus X$ ($Z' = \overline{X}' \setminus X'$) is a simple normal crossing divisor and let us assume given a decomposition $\{Z_i\}_{i=1}^r$ ($\{Z'_i\}_{i=1}^r$) of Z (Z'). Let $\Sigma = \prod_{i=1}^r \Sigma_i$ be a subset of \mathbb{Z}_p^r and let $f : (\overline{X}', Z') \longrightarrow (\overline{X}, Z)$ be a morphism. Then:*

- (1) *If f is a surjective strict smooth morphism with $f^*Z_i = Z'_i$ ($1 \leq i \leq r$), an object $\mathcal{E} \in \text{Isoc}^{\log}(\overline{X}, Z)$ is in $\text{Isoc}^{\log}(\overline{X}, Z)_{\Sigma(-\text{ss})}$ if and only if $f^*\mathcal{E}$ is in $\text{Isoc}^{\log}(\overline{X}', Z')_{\Sigma(-\text{ss})}$.*
- (2) *If f is log smooth and if there exists some $n := (n_i)_{i=1}^r$ with n_i 's prime to p such that $f^*Z_i = n_i Z'_i$ ($1 \leq i \leq r$) etale locally on \overline{X} and \overline{X}' , the pull-back $f^*\mathcal{E}$ of an object \mathcal{E} in $\text{Isoc}^{\log}(\overline{X}, Z)_{\Sigma(-\text{ss})}$ is contained in $\text{Isoc}^{\log}(\overline{X}', Z')_{n\Sigma(-\text{ss})}$.*

Proof. First let us prove (1). By Proposition 1.21(1), we may replace $\overline{X}, \overline{X}'$ by $\overline{X} \setminus Z_{\text{sing}}, f^{-1}(\overline{X} \setminus Z_{\text{sing}})$, respectively. (Then Z, Z' will be smooth.) Then, for an open covering $\overline{X} = \bigcup_\beta \overline{X}_\beta$ by finite number of open subschemes, we may replace $\overline{X}, \overline{X}'$ by $\coprod_\beta \overline{X}_\beta, \coprod_\beta f^{-1}(\overline{X}_\beta)$ by Proposition 1.21(2). Then, to prove the assertion in this situation, we may replace $\overline{X}, \overline{X}'$ by $\overline{X}_\beta, f^{-1}(\overline{X}_\beta)$. So we may assume that \overline{X} is affine connected, Z is a connected smooth divisor in \overline{X} and $r = 1$ (and so $Z = Z_1$). Moreover, we may assume that there exists a charted smooth standard small frame $((X, \overline{X}, \overline{\mathcal{X}}), t)$ enclosing (X, \overline{X}) . Moreover, by taking an open covering $\overline{X}' = \bigcup_\beta \overline{X}'_\beta$ by finite number of affine opens and replacing \overline{X}' by $\coprod_\beta \overline{X}'_\beta$, we may assume that \overline{X}' is a disjoint union of connected affine schemes \overline{X}'_β . Then, by Lemma 1.12 and Remark 1.13, there exists a morphism of charted smooth standard small frames $\tilde{f}_\beta : ((\overline{X}'_\beta \setminus Z', \overline{X}'_\beta, \overline{\mathcal{X}}_\beta), t'_\beta) \longrightarrow ((X, \overline{X}, \overline{\mathcal{X}}), t)$ compatible with the composite $(\overline{X}'_\beta \setminus Z', \overline{X}'_\beta) \hookrightarrow (X', \overline{X}') \xrightarrow{f} (X, \overline{X})$ such that $\tilde{f} : \overline{\mathcal{X}}'_\beta \longrightarrow \overline{\mathcal{X}}$ is smooth and that $\tilde{f}^*t = t'_\beta$. Then \mathcal{E} induces the log- ∇ -module (E, ∇) on $\overline{\mathcal{X}}_K$ with respect to t , $f^*\mathcal{E}$ induces the log- ∇ -module $(E'_\beta, \nabla'_\beta)$ on $\overline{\mathcal{X}}'_{\beta,K}$ with respect to t'_β for each β and we have $\tilde{f}_{\beta,K}^*(E, \nabla) = (E'_\beta, \nabla'_\beta)$, where $\tilde{f}_{\beta,K} : \overline{\mathcal{X}}'_{\beta,K} \longrightarrow \overline{\mathcal{X}}_K$ is the morphism induced by \tilde{f}_β . By definition, \mathcal{E} has exponents in Σ (with semisimple residues) if and only if the exponents of (E, ∇) along $\{t = 0\}$ is contained in Σ (and there exists some $P(x) \in \mathbb{Z}_p[x]$ without multiple roots with $P(\text{res}) = 0$, where res is the residue of (E, ∇) along $\{t = 0\}$) and $f^*\mathcal{E}$ has exponents in Σ (with semisimple residues) if

and only if, for any β , the exponents of $(E'_\beta, \nabla'_\beta)$ along $\{t'_\beta = 0\}$ is contained in Σ (and there exists some $P_\beta(x) \in \mathbb{Z}_p[x]$ without multiple roots with $P_\beta(\text{res}'_\beta) = 0$, where res'_β is the residue of $(E'_\beta, \nabla'_\beta)$ along $\{t'_\beta = 0\}$). Note now that, since $\coprod_\beta \tilde{f}_\beta$ is surjective and smooth, the induced map

$$\text{End}(E|_{\{t=0\}}) \longrightarrow \prod_\beta \text{End}(E'_\beta|_{\{t'_\beta=0\}}),$$

which sends res to $(\text{res}'_\beta)_\beta$ is injective. From this injectivity, we see that, for any $P(x) \in \mathbb{Z}_p[x]$, we have $P(\text{res}) = 0$ if and only if $P(\text{res}_\beta) = 0$ for any β . So the exponents of (E, ∇) along $\{t = 0\}$ is contained in Σ (and there exists some $P(x) \in \mathbb{Z}_p[x]$ without multiple roots with $P(\text{res}) = 0$) if and only if the exponents of $(E'_\beta, \nabla'_\beta)$ along \mathcal{Z}_β is contained in Σ (and there exists some $P_\beta(x) \in \mathbb{Z}_p[x]$ without multiple roots with $P_\beta(\text{res}'_\beta) = 0$) for any β . So \mathcal{E} has exponents in Σ (with semisimple residues) if and only if $f^*\mathcal{E}$ has exponents in Σ (with semisimple residues). Hence we have proved the assertion (1).

Next we prove (2). By the argument as in the proof of (1), we may assume that \overline{X} is affine connected, Z is a connected smooth divisor in \overline{X} and $r = 1$ (and so $Z = Z_1$). By using Proposition 1.21(1), we may assume that Z' is smooth, and by replacing \overline{X}' by its affine open subschemes, we may assume \overline{X}' is affine connected and Z' is a connected smooth divisor. Since we may work etale locally on \overline{X} and \overline{X}' by (1), we may assume that Z, Z' are defined as the zero locus of \bar{t}, \bar{t}' with $f^*\bar{t} = \bar{t}'^n$. Also, we may assume that there exists a charted smooth standard small frame $((X, \overline{X}, \overline{\mathcal{X}}), t)$ enclosing (X, \overline{X}) such that t lifts \bar{t} . Then, by Lemma 1.12, there exists a charted smooth standard small frame $((X', \overline{X}', \overline{\mathcal{X}}'), t')$ enclosing (X', \overline{X}') such that $t' \in \Gamma(\overline{\mathcal{X}}', \mathcal{O}_{\overline{\mathcal{X}}'})$ is a lift of \bar{t}' and a morphism $\tilde{f} : ((X', \overline{X}', \overline{\mathcal{X}}'), t') \longrightarrow ((X, \overline{X}, \overline{\mathcal{X}}), t)$ with $\tilde{f}^*t = t'^n$ which is compatible with $f : (X', \overline{X}') \longrightarrow (X, \overline{X})$. Then \mathcal{E} induces the log- ∇ -module (E, ∇) on $\overline{\mathcal{X}}_K$ with respect to t , $f^*\mathcal{E}$ induces the log- ∇ -module (E', ∇') on $\overline{\mathcal{X}}'_K$ with respect to t' and we have $\tilde{f}_K^*(E, \nabla) = (E', \nabla')$, where $\tilde{f}_K : \overline{\mathcal{X}}'_K \longrightarrow \overline{\mathcal{X}}_K$ is the morphism induced by \tilde{f} . When \mathcal{E} has exponents in Σ (with semisimple residues), the exponents of (E, ∇) along $\{t = 0\}$ is contained in Σ (and there exists some $P(x) \in \mathbb{Z}_p[x]$ without multiple roots with $P(\text{res}) = 0$, where res is the residue of (E, ∇) along $\{t = 0\}$). Then, by the equality $\tilde{f}^*t = t'^n$, we see that the residue res' of (E', ∇') along $\{t' = 0\}$ satisfies $\text{res}' = n\tilde{f}_K^*(\text{res})$. Hence the exponents of (E', ∇') along $\{t' = 0\}$ is contained in $n\Sigma$ (and we have $P(\text{res}'/n) = 0$). Hence $f^*\mathcal{E}$ has exponents in $n\Sigma$ (with semisimple residues) and so we have proven the assertion (2). \square

2 Convergent isocrystals on stacks

In this section, we give a definition of the category of (log) convergent isocrystals on (fine log) algebraic stacks and prove the equivalences (0.7), (0.8) and (0.9).

2.1 Basic definitions

In this subsection, we give a definition of the category of (log) convergent isocrystals on (fine log) algebraic stacks. For an algebraic stack X of finite type over k , we define the category Sch/X as follows: An object in Sch/X is an object Y in Sch endowed with a 1-morphism $a_Y : Y \rightarrow X$. A morphism $(Y, a_Y) \rightarrow (Y', a_{Y'})$ in Sch/X is a morphism of schemes $\varphi : Y \rightarrow Y'$ endowed with a 2-isomorphism $\tau : a_{Y'} \circ \varphi \xrightarrow{\cong} a_Y$. (Note that, when X is in Sch , the category Sch/X is nothing but the category of objects in Sch over X .) Using this category, we define the notion of convergent isocrystals on algebraic stacks as follows:

Definition 2.1. *Let X be an algebraic stack of finite type over k . We define the category $\text{Isoc}(X)$ of convergent isocrystals on X over K (resp. the category $F\text{-Isoc}(X)$ of convergent F -isocrystals on X over K , the category $F\text{-Isoc}(X)^\circ$ of unit-root convergent F -isocrystals on X over K) as the category of pairs*

$$(\{\mathcal{E}_Y\}_{Y \in \text{Ob}(\text{Sch}/X)}, \{\varphi_{\mathcal{E}}\}_{\varphi: Y \rightarrow Y' \in \text{Mor}(\text{Sch}/X)}),$$

where $\mathcal{E}_Y \in \text{Isoc}(Y)$ (resp. $\mathcal{E}_Y \in F\text{-Isoc}(Y)$, $\mathcal{E}_Y \in F\text{-Isoc}(Y)^\circ$) and $\varphi_{\mathcal{E}}$ is an isomorphism $\varphi^* \mathcal{E}_{Y'} \xrightarrow{\cong} \mathcal{E}_Y$ in $\text{Isoc}(Y)$ (resp. $F\text{-Isoc}(Y)$, $F\text{-Isoc}(Y)^\circ$) satisfying the cocycle condition $\varphi_{\mathcal{E}} \circ \varphi'^* \varphi'_{\mathcal{E}} = (\varphi' \circ \varphi)_{\mathcal{E}}$ for $Y \xrightarrow{\varphi} Y' \xrightarrow{\varphi'} Y''$ in Sch/X .

For a scheme X in Sch , the above definition of the categories $\text{Isoc}(X)$, $F\text{-Isoc}(X)$, $F\text{-Isoc}(X)^\circ$ coincides with the usual definition: The equivalence is given by

$$\begin{aligned} \mathcal{E} &\mapsto (\{a_Y^* \mathcal{E}\}_{(Y, a_Y) \in \text{Ob}(\text{Sch}/X)}, \{\varphi_{\mathcal{E}} : \varphi^* a_{Y'}^* \mathcal{E} \xrightarrow{\cong} a_Y^* \mathcal{E}\}_{\varphi: Y \rightarrow Y' \in \text{Mor}(\text{Sch}/X)}), \\ (\{\mathcal{E}_Y\}_{(Y, a_Y) \in \text{Ob}(\text{Sch}/X)}, \{\varphi_{\mathcal{E}}\}_{\varphi: Y \rightarrow Y' \in \text{Mor}(\text{Sch}/X)}) &\mapsto \mathcal{E}_X. \end{aligned}$$

Let X be a Deligne-Mumford stack of finite type over k , let $\epsilon : X_0 \rightarrow X$ be an etale surjective morphism from a scheme X_0 in Sch and let X_n ($n = 0, 1, 2$) be the $(n+1)$ -fold fiber product of X_0 over X . Then we have a 2-truncated simplicial scheme X_\bullet endowed with a morphism $X_\bullet \rightarrow X$ and we have canonical functors

$$(2.1) \quad \begin{aligned} \text{Isoc}(X) &\longrightarrow \text{Isoc}(X_\bullet), \quad F\text{-Isoc}(X) \longrightarrow F\text{-Isoc}(X_\bullet), \\ F\text{-Isoc}(X)^\circ &\longrightarrow F\text{-Isoc}(X_\bullet)^\circ. \end{aligned}$$

Then we have the following:

Proposition 2.2. *Let the notations be as above. Then the functors (2.1) are equivalences.*

Proof. We treat only the case for $\text{Isoc}(X)$. (The other cases can be proven in the same way.) We define the inverse functor as follows: Given an object $\mathcal{E}_\bullet \in \text{Isoc}(X_\bullet)$ and $Y \rightarrow X$ in Sch/X , let us put $Y_\bullet := X_\bullet \times_X Y$. Then \mathcal{E}_\bullet naturally induces an object in $\text{Isoc}(Y_\bullet)$, which we denote by \mathcal{E}'_\bullet . Then, since $Y_\bullet \rightarrow Y$ is an etale Čech hypercovering of schemes, we have the equivalence $\text{Isoc}(Y) \xrightarrow{\cong} \text{Isoc}(Y_\bullet)$ by [35, 4.4].

So \mathcal{E}'_\bullet induces an object \mathcal{E}_Y in $\text{Isoc}(Y)$. Then $\{\mathcal{E}_Y\}_{Y \in \text{Sch}/X}$ forms in a natural way an object in $\text{Isoc}(X)$. One can prove that this functor gives the desired inverse functor by using the étale descent property for the category of convergent isocrystals [35, 4.4] again. \square

Example 2.3. Let X be an object in Sch and let G be a finite étale group scheme over k acting on X . Then we have the quotient stack $[X/G]$, which is a Deligne-Mumford stack. $X_0 := X \rightarrow [X/G]$ is a surjective finite étale morphism and if we denote the $(n+1)$ -fold fiber product of X over $[X/G]$ by X_n ($n = 0, 1, 2$), we have $X_n \cong X \times_k G^n$. In this case, an object in $\text{Isoc}([X/G]) \cong \text{Isoc}(X_\bullet)$ is nothing but a convergent isocrystal \mathcal{E} on X endowed with an equivariant action of G .

In the following, we generalize the above definition of the category of convergent (F -)isocrystals on algebraic stacks to the case with log structures. First recall that the notion of a fine log structure on an algebraic stack X is defined in [34, 5.1] as a sheaf of monoids M_X on the lisse-étale site $X_{\text{lis-ét}}$ endowed with a monoid homomorphism $M_X \rightarrow \mathcal{O}_{X_{\text{lis-ét}}}$ such that, for any object U in $X_{\text{lis-ét}}$, the log structure $M_X|_{U_{\text{ét}}}$ on $U_{\text{ét}}$ is fine and that, for any morphism $\varphi : U' \rightarrow U$ in $X_{\text{lis-ét}}$, the map $\varphi^*(M_X|_{U_{\text{ét}}}) \rightarrow M_X|_{U'_{\text{ét}}}$ is an isomorphism. We call a pair (X, M_X) of an algebraic stack X and a fine log structure M_X on it a fine log algebraic stack. For a fine log algebraic stack (X, M_X) and a morphism of algebraic stacks $f : Y \rightarrow X$, one can define the pull-back log structure f^*M_X on $Y_{\text{lis-ét}}$ (see [34, p.773]). When Y is a scheme, the restriction $f^*M_X|_{Y_{\text{ét}}}$ of f^*M_X to the étale site $Y_{\text{ét}}$ gives a fine log structure on Y (see [34, 5.3]). Now we define the notion of locally free log convergent isocrystals on fine log algebraic stacks as follows:

Definition 2.4. Let (X, M_X) be a fine log algebraic stack of finite type over k . We define the category $\text{Isoc}^{\log}(X, M_X)$ of locally free log convergent isocrystals on (X, M_X) over K (resp. the category $F\text{-Isoc}^{\log}(X, M_X)$ of locally free log convergent F -isocrystals on (X, M_X) over K) as the category of pairs

$$(\{\mathcal{E}_Y\}_{(Y, a_Y) \in \text{Ob}(\text{Sch}/X)}, \{\varphi_{\mathcal{E}}\}_{\varphi : Y \rightarrow Y' \in \text{Mor}(\text{Sch}/X)}),$$

where $\mathcal{E}_Y \in \text{Isoc}^{\log}(Y, a_Y^*M_X|_{Y_{\text{ét}}})$ (resp. $\mathcal{E}_Y \in F\text{-Isoc}^{\log}(Y, a_Y^*M_X|_{Y_{\text{ét}}})$) and $\varphi_{\mathcal{E}}$ is an isomorphism $\varphi^*\mathcal{E}_{Y'} \xrightarrow{\cong} \mathcal{E}_Y$ in $\text{Isoc}^{\log}(Y, a_Y^*M_X|_{Y_{\text{ét}}})$ (resp. $F\text{-Isoc}^{\log}(Y, a_Y^*M_X|_{Y_{\text{ét}}})$) satisfying the cocycle condition $\varphi_{\mathcal{E}} \circ \varphi'^*\varphi'_{\mathcal{E}} = (\varphi' \circ \varphi)_{\mathcal{E}}$ for $Y \xrightarrow{\varphi} Y' \xrightarrow{\varphi'} Y''$ in Sch/X .

Note that, for $(Y, M_Y) \in \text{LSch}$ and a strict étale Čech hypercovering $(Y_\bullet, M_{Y_\bullet}) \rightarrow (Y, M_Y)$, we can prove the equivalence of categories $\text{Isoc}^{\log}(Y, M_Y) \xrightarrow{\cong} \text{Isoc}^{\log}(Y_\bullet, M_{Y_\bullet})$ in the same way as [35, 4.4]. (See also [41, 5.1.7].) So we can prove the following proposition in the same way as Proposition 2.2 (so we omit the proof):

Proposition 2.5. Let (X, M_X) be a fine log Deligne-Mumford stack of finite type over k , let $\epsilon : X_0 \rightarrow X$ be an étale surjective morphism from a scheme X_0 in Sch .

Let X_n ($n = 0, 1, 2$) be the $(n + 1)$ -fold fiber product of X_0 over X and let M_{X_n} be the restriction of M_X to $X_{n,\text{et}}$. Then the canonical functors

$$\text{Isoc}^{\log}(X, M_X) \longrightarrow \text{Isoc}^{\log}(X_{\bullet}, M_{X_{\bullet}}), \quad F\text{-Isoc}^{\log}(X, M_X) \longrightarrow F\text{-Isoc}^{\log}(X_{\bullet}, M_{X_{\bullet}})$$

are equivalences.

Once we define the categories $\text{Isoc}(X)$, $F\text{-Isoc}(X)$, $F\text{-Isoc}(X)^{\circ}$ (resp. $\text{Isoc}^{\log}(X, M_X)$, $F\text{-Isoc}^{\log}(X, M_X)$) for algebraic stacks X (resp. fine log algebraic stacks (X, M_X)), we can generalize the definition to the case of diagram of algebraic stacks (resp. diagram of fine log algebraic stacks) as follows.

Definition 2.6. Let Sta (resp. LSta) be the 2-category of algebraic stacks (resp. fine log algebraic stacks) of finite type over k . Then, for a category \mathcal{C} (regarded also as a discrete 2-category) and a 2-functor $\Phi : \mathcal{C} \longrightarrow \text{Sta}$ (resp. $\Phi : \mathcal{C} \longrightarrow \text{LSta}$), we define the category $\text{Isoc}(\Phi)$, $F\text{-Isoc}(\Phi)$, $F\text{-Isoc}(\Phi)^{\circ}$ (resp. $\text{Isoc}^{\log}(\Phi)$, $F\text{-Isoc}^{\log}(\Phi)$) as the category of pairs

$$(\{\mathcal{E}_Y\}_{Y \in \text{Ob}(\mathcal{C})}, \{\varphi_{\mathcal{E}}\}_{\varphi: Y \rightarrow Y' \in \text{Mor}(\mathcal{C})}),$$

where \mathcal{E}_Y is an object in $\text{Isoc}(\Phi(Y))$, $F\text{-Isoc}(\Phi(Y))$, $F\text{-Isoc}(\Phi(Y))^{\circ}$ (resp. $\text{Isoc}^{\log}(\Phi(Y))$, $F\text{-Isoc}^{\log}(\Phi(Y))$) and $\varphi_{\mathcal{E}}$ is an isomorphism $\Phi(\varphi)^* \mathcal{E}_{Y'} \xrightarrow{\cong} \mathcal{E}_Y$ satisfying the cocycle condition $\varphi_{\mathcal{E}} \circ \Phi(\varphi)^* \varphi'_{\mathcal{E}} = (\varphi' \circ \varphi)_{\mathcal{E}}$ for $Y \xrightarrow{\varphi} Y' \xrightarrow{\varphi'} Y''$ in \mathcal{C} .

In particular, we can define the categories $\text{Isoc}(X_{\bullet})$, $F\text{-Isoc}(X_{\bullet})$, $F\text{-Isoc}(X_{\bullet})^{\circ}$ (resp. $\text{Isoc}^{\log}(X_{\bullet}, M_{X_{\bullet}})$, $F\text{-Isoc}^{\log}(X_{\bullet}, M_{X_{\bullet}})$) for 2-truncated simplicial algebraic stacks X_{\bullet} (resp. 2-truncated simplicial fine log algebraic stacks $(X_{\bullet}, M_{X_{\bullet}})$). It is easy to see that, for $X \in \text{Sta}$ and a 2-truncated etale Čech hypercovering $X_{\bullet} \longrightarrow X$, we have the equivalences

$$\text{Isoc}(X) \xrightarrow{\cong} \text{Isoc}(X_{\bullet}), \quad F\text{-Isoc}(X) \xrightarrow{\cong} F\text{-Isoc}(X_{\bullet}), \quad F\text{-Isoc}(X)^{\circ} \xrightarrow{\cong} F\text{-Isoc}(X_{\bullet})^{\circ}$$

and for $X \in \text{LSta}$ and a 2-truncated strict etale Čech hypercovering $(X_{\bullet}, M_{X_{\bullet}}) \longrightarrow (X, M_X)$, we have the equivalences

$$\text{Isoc}^{\log}(X, M_X) \xrightarrow{\cong} \text{Isoc}^{\log}(X_{\bullet}, M_{X_{\bullet}}), \quad F\text{-Isoc}^{\log}(X, M_X) \xrightarrow{\cong} F\text{-Isoc}^{\log}(X_{\bullet}, M_{X_{\bullet}}).$$

Next we give a definition of the category of locally free log convergent isocrystals on certain algebraic stacks ‘with exponent condition’. Let (X, M_X) be a fine log algebraic stack of finite type over k satisfying the following condition (*):

(*) There exists a smooth surjective morphism $a_Y : Y \longrightarrow X$ with $Y \in \text{Sch}$ such that Y is smooth over k and that $M_Y := a_Y^* M_X|_{Y_{\text{et}}}$ is associated to a simple normal crossing divisor Z on Y .

Under this condition, we define the notion of a decomposition of M_X as follows:

Definition 2.7. Let (X, M_X) be as above. Then a family of sub fine log structures $\{M_{X,i}\}_{i=1}^r$ of M_X is called a decomposition of M_X (with respect to $a_Y : Y \rightarrow X$) if the log structure $M_{Y,i} := a_Y^* M_{X,i}|_{Y_{\text{et}}}$ is associated to a simple normal crossing subdivisor Z_i of Z ($1 \leq i \leq r$) with $Z = \sum_{i=1}^r Z_i$.

Concerning the above definition, we have the following independence result.

Lemma 2.8. Let (X, M_X) be as above and let $\{M_{X,i}\}_{i=1}^r$ be a family of sub fine log structures of M_X . Then, if $\{M_{X,i}\}_{i=1}^r$ is a decomposition of M_X with respect to one morphism $a_Y : Y \rightarrow X$ as in (*), it is a decomposition of M_X with respect to any morphism $a_Y : Y \rightarrow X$ as in (*).

Proof. Assume that $\{M_{X,i}\}_{i=1}^r$ is a decomposition of M_X with respect to a morphism $a_Y : Y \rightarrow X$ as in (*) and let us take another morphism $a_{Y'} : Y' \rightarrow X$ as in (*). Then, by taking a suitable surjective etale morphism $f : Y'' \rightarrow Y \times_X Y'$, we see that the composite $a_{Y''} : Y'' \xrightarrow{f} Y \times_X Y' \rightarrow X$ also satisfies the condition (*). Also, the morphisms $Y'' \rightarrow Y, Y'' \rightarrow Y'$ induced by projection are smooth. For $m = 0, 1, 2$, let Y''_m be the $(m+1)$ -fold fiber product of Y'' over Y' . Let $\pi_j : Y''_1 \rightarrow Y''_0 = Y''$ ($j = 0, 1$) be the projections and let $a_{Y''_m}$ be the natural morphism $Y''_m \rightarrow X$ induced by (any) projection $Y''_m \rightarrow Y''$ and $a_{Y''}$.

By assumption, $a_Y^* M_X|_{Y_{\text{et}}}$ is associated to a simple normal crossing divisor Z on Y and $a_Y^* M_{X,i}|_{Y_{\text{et}}}$ is associated to a simple normal crossing subdivisor Z_i of Z with $Z = \sum_{i=1}^r Z_i$. By [34, 5.3], we see that the log structure $a_Y^* M_X$ (resp. $a_Y^* M_{X,i}$) on $Y_{\text{lis-et}}$ is also associated to Z (resp. Z_i). So, if we define Z'' (resp. Z''_i) to be the pull-back of Z (resp. Z_i) to Y'' , we see that the log structure $a_{Y''}^* M_X$ (resp. $a_{Y''}^* M_{X,i}$) is associated to Z'' (resp. Z''_i), and by pulling it back by π_j , we see that the log structure $a_{Y''_1}^* M_X$ (resp. $a_{Y''_1}^* M_{X,i}$) is associated to $\pi_0^* Z''$ (resp. $\pi_0^* Z''_i$) and also to $\pi_1^* Z''$ (resp. $\pi_1^* Z''_i$). So we have the equality $\pi_0^* Z'' = \pi_1^* Z''$ (resp. $\pi_0^* Z''_i = \pi_1^* Z''_i$) and it satisfies the cocycle condition on Y''_2 . So there exists a normal crossing divisor Z' (resp. Z'_i) on Y' which is pulled back to Z'' (resp. Z''_i) by the morphism $Y'' \rightarrow Y'$. (Then we see also the equality $Z' = \sum_{i=1}^r Z'_i$.) Let N (resp. N_i) be the log structure on Y'_{et} associated to Z' (resp. Z'_i). Then N and $M_{Y'} := a_{Y'}^* M_X|_{Y'_{\text{et}}}$ (resp. N_i and $M_{Y',i} := a_{Y'}^* M_{X,i}|_{Y'_{\text{et}}}$) coincide on $Y'_{\bullet,\text{et}}$. So they are equal by [34, Theorem A.1]. Hence $M_{Y'}$ (resp. $M_{Y',i}$) is associated to Z' (resp. Z'_i). So, by the condition (*), we see that Z' (which is a priori a normal crossing divisor) is in fact a simple normal crossing divisor and so are Z'_i 's. Hence $\{M_{X,i}\}_{i=1}^r$ is also a decomposition of M_X with respect to the morphism $a_{Y'} : Y' \rightarrow X$. So we are done. \square

Now we define the category of locally free log convergent isocrystals with exponents in Σ for certain fine log algebraic stacks, as follows:

Definition 2.9. Let (X, M_X) be a fine log algebraic stack of finite type over k satisfying the condition (*) above and let $\{M_{X,i}\}_{i=1}^r$ be a decomposition of M_X . Let us take $\Sigma = \prod_{i=1}^r \Sigma_i \subseteq \mathbb{Z}_p^r$ and let us take $a_Y : Y \rightarrow X, M_Y, Z = \sum_{i=1}^r Z_i$ as in (*) and Definition 2.7. Then we say that a locally free log convergent isocrystal \mathcal{E} on (X, M_X)

over K has exponents in Σ (resp. has exponents in Σ with semisimple residues) with respect to the decomposition $\{M_{X,i}\}_{i=1}^r$ if the object $\mathcal{E}_Y \in \text{Isoc}^{\log}(Y, M_Y)$ induced by \mathcal{E} has exponents in Σ (resp. has exponents in Σ with semisimple residues) with respect to the decomposition $\{Z_i\}_{i=1}^r$ of Z . We denote the category of locally free log convergent isocrystals on (X, M_X) having exponents in Σ (resp. having exponents in Σ with semisimple residues) by $\text{Isoc}^{\log}(X, M_X)_{\Sigma}$ (resp. $\text{Isoc}^{\log}(X, M_X)_{\Sigma\text{-ss}}$).

We have the following independence result.

Lemma 2.10. *The above definition is independent of the choice of $a_Y : Y \rightarrow X$ as in (*).*

Proof. Let $\mathcal{E} \in \text{Isoc}^{\log}(X, M_X)$ and let us take two morphisms $a_Y : Y \rightarrow X$ $a_{Y'} : Y' \rightarrow X$ as in (*). Then, by taking a suitable surjective étale morphism $f : Y'' \rightarrow Y \times_X Y'$, we see that the composite $a_{Y''} : Y'' \xrightarrow{f} Y \times_X Y' \rightarrow X$ also satisfies the condition (*), and the morphisms $Y'' \rightarrow Y, Y'' \rightarrow Y'$ induced by projection are surjective and smooth. So, it suffices to prove the following: For two morphism $a_Y : Y \rightarrow X, a_{Y'} : Y' \rightarrow X$ as in (*) and a surjective smooth morphism $f : Y' \rightarrow Y$ with $a_Y \circ f$ 2-isomorphic to $a_{Y'}$, \mathcal{E} has exponents in Σ (with semisimple residues) for the morphism $a_Y : Y \rightarrow X$ if and only if so does it for the morphism $a_{Y'} : Y' \rightarrow X$. If we take $Z = \sum_{i=1}^r Z_i$ as in (*) and Definition 2.7 and if we put $Z' := f^*Z, Z'_i := f^*Z_i$, the claim we should prove is rewritten as follows: $a_Y^* \mathcal{E}$ has exponents in Σ (with semisimple residues) with respect to the decomposition $\{Z_i\}_i$ of Z if and only if $a_{Y'}^* \mathcal{E} \cong f^*(a_Y^* \mathcal{E})$ has exponents in Σ (with semisimple residues) with respect to the decomposition $\{Z'_i\}$ of Z' . This follows from Proposition 1.22(1). So we are done. \square

We can also define the category of locally free log convergent isocrystals with exponent condition for certain diagram of fine log algebraic stacks.

Definition 2.11. *Let $(X_{\bullet}, M_{X_{\bullet}})$ be a diagram of fine log algebraic stacks of finite type over k indexed by a small category \mathcal{C} such that each (X_c, M_{X_c}) ($c \in \mathcal{C}$) satisfies the condition (*) and let $\{M_{X_{\bullet},i}\}_{i=1}^r$ be a family of sub fine log structures of $M_{X_{\bullet}}$ such that, for any $c \in \mathcal{C}$, the induced family $\{M_{X_c,i}\}_{i=1}^r$ gives a decomposition of M_{X_c} . (We call such a family $\{M_{X_{\bullet},i}\}_{i=1}^r$ a decomposition of $M_{X_{\bullet}}$.) Then we say that an object \mathcal{E}_{\bullet} in $\text{Isoc}^{\log}(X_{\bullet}, M_{X_{\bullet}})$ has exponents in Σ (resp. has exponents in Σ with semisimple residues) with respect to the decomposition $\{M_{X_{\bullet},i}\}_{i=1}^r$ if, for any $c \in \mathcal{C}$, the object $\mathcal{E}_c \in \text{Isoc}^{\log}(X_c, M_{X_c})$ induced by \mathcal{E}_{\bullet} is contained in $\text{Isoc}^{\log}(X_c, M_{X_c})_{\Sigma}$ (resp. $\text{Isoc}^{\log}(X_c, M_{X_c})_{\Sigma\text{-ss}}$). We denote the category of objects in $\text{Isoc}^{\log}(X_{\bullet}, M_{X_{\bullet}})$ having exponents in Σ (resp. having exponents in Σ with semisimple residues) with respect to the decomposition $\{M_{X_{\bullet},i}\}_{i=1}^r$ by $\text{Isoc}^{\log}(X_{\bullet}, M_{X_{\bullet}})_{\Sigma}$ (resp. $\text{Isoc}^{\log}(X_{\bullet}, M_{X_{\bullet}})_{\Sigma\text{-ss}}$).*

We have a functoriality property of the category of the form $\text{Isoc}^{\log}(X, M_X)_{\Sigma(-\text{ss})}$ for certain morphism of fine log algebraic stacks.

Proposition 2.12. *Let (X, M_X) be a fine log algebraic stack of finite type over k satisfying the condition $(*)$ and let $\{M_{X,i}\}_{i=1}^r$ be a decomposition of M_X . Let us assume given another fine log algebraic stack $(X', M_{X'})$ of finite type over k and a strict smooth morphism $f : (X', M_{X'}) \rightarrow (X, M_X)$ over k . Then, $(X', M_{X'})$ also satisfies the condition $(*)$ and $\{f^*M_{X,i}\}_{i=1}^r$ gives a decomposition of $M_{X'}$. Moreover, for a subset $\Sigma = \prod_{i=1}^r \Sigma_i$ in \mathbb{Z}_p^r , f induces the functor $f^* : \text{Isoc}^{\log}(X, M_X)_{\Sigma(-ss)} \rightarrow \text{Isoc}^{\log}(X', M_{X'})_{\Sigma(-ss)}$.*

Proof. Let us take $a_Y : Y \rightarrow X, Z = \sum_i Z_i$ as in $(*)$. Then we can take a diagram $a_{Y'} : Y' \rightarrow Y \times_X X' \rightarrow X'$ for some $Y' \in \text{Sch}$ such that the first map is surjective etale. Denote the composite $Y' \rightarrow Y \times_X X' \xrightarrow{\text{proj.}} Y$ by b . Then $a_{Y'}$ is surjective smooth and the log structure $a_{Y'}^* M_{X'}$ is equal to $a_Y^* f^* M_X \cong b^* a_Y^* M_X$, which is equal to the log structure associated to the simple normal crossing divisor $b^{-1}(Z)$ in Y' . So $(X', M_{X'})$ satisfies the condition $(*)$. Moreover, since the log structure $a_{Y'}^* f^* M_{X,i}|_{Y'_{\text{et}}} \cong b^* a_Y^* M_{X,i}|_{Y_{\text{et}}}$ is associated to the simple normal crossing subdivisor $b^{-1}(Z_i)$ of $b^{-1}(Z)$ in Y' , we see that $\{f^*M_{X,i}\}_{i=1}^r$ gives a decomposition of $M_{X'}$.

Let us prove the last assertion. Let us take an object \mathcal{E} in $\text{Isoc}^{\log}(X, M_X)_{\Sigma(-ss)}$. Then the object \mathcal{E}_Y in $\text{Isoc}^{\log}(Y, Z)$ induced by \mathcal{E} is contained in $\text{Isoc}^{\log}(Y, Z)_{\Sigma(-ss)}$. Since the morphism $b : (Y', b^{-1}(Z)) \rightarrow (Y, Z)$ is strict smooth, $b^*(\mathcal{E}_Y) \cong (f^*\mathcal{E})_{Y'} (:= \text{the object in } \text{Isoc}^{\log}(Y', b^{-1}(Z)) \text{ induced by } f^*\mathcal{E})$ is contained in the category $\text{Isoc}^{\log}(Y', b^{-1}(Z))_{\Sigma(-ss)}$ by Proposition 1.22(1). Hence we have $f^*\mathcal{E} \in \text{Isoc}^{\log}(X', M_{X'})_{\Sigma(-ss)}$, as desired. \square

Corollary 2.13. *Let (X, M_X) be a fine log algebraic stack of finite type over k satisfying the condition $(*)$ and let $(X_{\bullet}, M_{X_{\bullet}}) \rightarrow (X, M_X)$ be a 2-truncated strict etale Čech hypercovering. Let $\{M_{X,i}\}_{i=1}^r$ be a decomposition of M_X , let $\{M_{X_{\bullet},i}\}$ be the induced decomposition of $M_{X_{\bullet}}$ and let $\Sigma = \prod_{i=1}^r \Sigma_i$ be a subset of \mathbb{Z}_p^r . Then we have an equivalence of categories*

$$(2.2) \quad \text{Isoc}^{\log}(X, M_X)_{\Sigma(-ss)} \xrightarrow{\cong} \text{Isoc}^{\log}(X_{\bullet}, M_{X_{\bullet}})_{\Sigma(-ss)}.$$

Proof. By Proposition 2.12 (see also Definition 2.11), we have the functor (2.2), and it is fully faithful because the functor $\text{Isoc}^{\log}(X, M_X) \rightarrow \text{Isoc}^{\log}(X_{\bullet}, M_{X_{\bullet}})$ is an equivalence. Let us prove the essential surjectivity. So let us take an object $\mathcal{E}_{\bullet} \in \text{Isoc}^{\log}(X_{\bullet}, M_{X_{\bullet}})_{\Sigma(-ss)}$ and take an object $\mathcal{E} \in \text{Isoc}^{\log}(X, M_X)$ which is sent to \mathcal{E}_{\bullet} . Then there exists a morphism $a_Y : Y \rightarrow X_0$ and $Z \subseteq Y$ as in $(*)$ for X_0 and the object $\mathcal{E}_{0,Y}$ in $\text{Isoc}^{\log}(Y, Z)$ induced by \mathcal{E}_0 is actually contained in $\text{Isoc}^{\log}(Y, Z)_{\Sigma(-ss)}$. Then the composite $Y \xrightarrow{a_Y} X_0 \rightarrow X$ and $Z \subseteq Y$ satisfy the condition $(*)$ for X and the object $\mathcal{E}_{0,Y}$ is equal to the object \mathcal{E}_Y in $\text{Isoc}^{\log}(Y, Z)$ induced by \mathcal{E} . Hence we have $\mathcal{E} \in \text{Isoc}^{\log}(X, M_X)_{\Sigma(-ss)}$, as desired. \square

Finally in this subsection, we give how to define fine log structures on Deligne-Mumford stacks and give examples which are useful in this paper.

Let X be a Deligne-Mumford stack of finite type over k , let $\epsilon : X_0 \rightarrow X$ be an etale surjective morphism from a scheme X_0 separated of finite type over k and

let X_n ($n = 0, 1, 2$) be the $(n + 1)$ -fold fiber product of X_0 over X . Assume that there exists a fine log structure M_{X_\bullet} on the 2-truncated simplicial scheme X_\bullet such that the transition maps $(X_a, M_{X_a}) \rightarrow (X_b, M_{X_b})$ ($a, b \in \{0, 1, 2\}$) are all strict. Then we can define the fine log structure M_X on X in the following way: For an object $Y \rightarrow X$ in $X_{\text{lis-et}}$, we have the 2-truncated etale Čech hypercovering $X_\bullet \times_X Y \rightarrow Y$. Then the pull-back $M_{X_\bullet \times_X Y}$ of M_{X_\bullet} to $X_\bullet \times_X Y$ descends to the fine log structure M_Y on Y_{et} and it is functorial with respect to Y . So $\{M_Y\}_Y$ defines a fine log structure on X .

Example 2.14. Let (X, M_X) be a connected Noetherian fs log scheme and let $(Y, M_Y) \rightarrow (X, M_X)$ be a finite Kummer log etale (finite log etale of Kummer type in the terminology in [31]) Galois covering with Galois group G . For $m = 0, 1, 2$, let (Y_m, M_{Y_m}) be the $(m + 1)$ -fold fiber product of (Y, M_Y) over (X, M_X) in the category of fs log schemes. Then we have $(Y_m, M_{Y_m}) \cong (Y_0, M_{Y_0}) \times G^m$ naturally (see [13]) and so $Y_\bullet \rightarrow [Y/G]$ gives an etale Čech hypercovering of $[Y/G]$. So the log structure M_{Y_\bullet} on Y_\bullet induces the log structure $M_{[Y/G]}$ on $[Y/G]$.

Example 2.15. Let X be a scheme smooth separated of finite type over k and let M_X be a log structure on X associated to a simple normal crossing divisor $Z = \bigcup_{i=1}^r Z_i$ (with each Z_i irreducible). Let $f : (Y, M_Y) \rightarrow (X, M_X)$ be a finite Kummer log etale Galois covering with Galois group G , let Y^{sm} be the smooth locus of Y and put $M_{Y^{\text{sm}}} := M_Y|_{Y^{\text{sm}}}$. Also, let X' be the image of Y^{sm} in X and put $M_{X'} := M_X|_{X'}$. Then Y^{sm} is G -stable and f induces the morphism $f' : (Y^{\text{sm}}, M_{Y^{\text{sm}}}) \rightarrow (X', M_{X'})$, which is again a finite Kummer log etale Galois covering with Galois group G . So, by Example 2.14, the log structure $M_{[Y^{\text{sm}}/G]}$ is defined. (By construction, we have $M_{[Y^{\text{sm}}/G]} = M_{[Y/G]}|_{[Y^{\text{sm}}/G]}$, where $M_{[Y/G]}$ is as in Example 2.14.) Note that, by [33, 5.2] and the perfectness of k , the log structure $M_{Y^{\text{sm}}}$ is associated to some normal crossing divisor Z' and the Kummer type assumption of f implies that Z' is in fact a simple normal crossing divisor. For any i , $(f'^*(X' \cap Z_i))_{\text{red}}$ defines a simple normal crossing subdivisor Z'_i of Z' which is G -stable and we have $Z' = \sum_{i=1}^r Z'_i$. Then, the log structure on $Y_\bullet^{\text{sm}} := X' \times_X Y_\bullet \cong Y^{\text{sm}} \times G^\bullet$ (where Y_\bullet is as in Example 2.14) associated to $Z'_i \times G^\bullet$ induces a sub log structure $M_{[Y^{\text{sm}}/G], i}$ of $M_{[Y^{\text{sm}}/G]}$ and the family $\{M_{[Y^{\text{sm}}/G], i}\}_{i=1}^r$ defines a decomposition of $M_{[Y^{\text{sm}}/G]}$.

2.2 First stacky equivalence

In this subsection, we give a proof of the equivalences (0.7) and (0.8). To do so, first let us recall the equivalence of Crew.

Let X be a connected smooth varieties over k . Crew proved in [9] the equivalence

$$(2.3) \quad G : \text{Rep}_{K^\sigma}(\pi_1(X)) \xrightarrow{=} F\text{-Isoc}(X)^\circ$$

(which is equal to (0.6)). Let us recall the definition of the functor (2.3), using the notion of convergent site which is introduced in [35]. Let ρ be an object

in $\text{Rep}_{K^\sigma}(\pi_1(X))$ and let $\mathcal{F} := \mathcal{F}_{0,\mathbb{Q}}$ be the corresponding object in $\text{Sm}_{K^\sigma}(X) = \text{Sm}_{O_K^\sigma}(X)_{\mathbb{Q}}$. (Here, for an additive category \mathcal{C} , $\mathcal{C}_{\mathbb{Q}}$ denotes the \mathbb{Q} -linearization of it and for an object A in \mathcal{C} , $A_{\mathbb{Q}}$ denotes the object A regarded as an object in $\mathcal{C}_{\mathbb{Q}}$.) For an enlargement (T, z_T) in the convergent site $(X/\text{Spf } O_K)_{\text{conv}}$ of X over $\text{Spf } O_K$ (that is, a p -adic formal scheme T of finite type over $\text{Spf } O_K$ and a morphism $z_T : T_0 := (T \otimes_{O_K} O_K/\mathfrak{m}_K)_{\text{red}} \rightarrow X$ over k), let $\mathcal{F}_{0,T}$ be the object in $\text{Sm}_{O_K^\sigma}(T)$ corresponding to $z_T^{-1}(\mathcal{F}_0)$ via the equivalence $\text{Sm}_{O_K^\sigma}(T) \cong \text{Sm}_{O_K^\sigma}(T_0)$ and let us define \mathcal{E}_T by $\mathcal{E}_T := (\mathcal{F}_{0,T} \otimes_{O_K^\sigma} \mathcal{O}_T)_{\mathbb{Q}}$ as an object in the \mathbb{Q} -linearized category of the category of coherent sheaves on T_{et} . Then $\mathcal{E} := \{\mathcal{E}_T\}_{(T, z_T)}$ defines a locally free convergent isocrystal on X over K . Moreover, the q -th power Frobenius endomorphism $F : X \rightarrow X$ induces the equivalence $F^{-1} : \text{Sm}_{O_K^\sigma}(X) \xrightarrow{\cong} \text{Sm}_{O_K^\sigma}(X)$ with $F^{-1}(\mathcal{F}_0) \cong \mathcal{F}_0$, and so we have the isomorphism

$$\Psi_T : (F^*\mathcal{E})_T = (F^{-1}(\mathcal{F}_0)_T \otimes_{O_K^\sigma} \mathcal{O}_T)_{\mathbb{Q}} \xrightarrow{\cong} (\mathcal{F}_{0,T} \otimes_{O_K^\sigma} \mathcal{O}_T)_{\mathbb{Q}} = \mathcal{E}_T$$

for any enlargement (T, z_T) , where $(F^*\mathcal{E})_T$ denotes the sheaf on T_{et} induced by the locally free convergent isocrystal $F^*\mathcal{E}$ and $F^{-1}(\mathcal{F}_0)_T$ is the object in $\text{Sm}_{O_K^\sigma}(T)$ corresponding to $z_T^{-1}(F^{-1}(\mathcal{F}_0))$ via the equivalence $\text{Sm}_{O_K^\sigma}(T) \cong \text{Sm}_{O_K^\sigma}(T_0)$. Then, if we put $\Psi := \{\Psi_T\}_{(T, z_T)}$, we see that the pair (\mathcal{E}, Ψ) defines an object in $F\text{-Isoc}(X)^\circ$, which we define to be the image of ρ by the functor G . In view of this description, we see that Crew's equivalence (2.3) is written as the equivalence

$$(2.4) \quad G : \text{Sm}_{K^\sigma}(X) \xrightarrow{\cong} F\text{-Isoc}(X)^\circ.$$

(The choice of a base point in the definition of $\pi_1(X)$ is not essential.) Also, we see easily the functoriality of the equivalence (2.4).

Now we proceed to prove the equivalences (0.7) and (0.8). In the following in this subsection, let $X \hookrightarrow \overline{X}$ be an open immersion of connected smooth varieties over k such that $\overline{X} \setminus X =: Z = \bigcup_{i=1}^r Z_i$ is a simple normal crossing divisor (with each Z_i irreducible). For $1 \leq i \leq r$, let v_i be the discrete valuation of $k(X)$ corresponding to the generic point of Z_i , let $k(X)_{v_i}$ be the completion of $k(X)$ with respect to v_i and let I_{v_i} be the inertia group of $k(X)_{v_i}$. (Then we have homomorphisms $I_{v_i} \rightarrow \pi_1(X)$ which are well-defined up to conjugacy.) Let us define $\text{Rep}_{K^\sigma}^{\text{fin}}(\pi_1(X))$ by

$$\text{Rep}_{K^\sigma}^{\text{fin}}(\pi_1(X)) := \{\rho \in \text{Rep}_{K^\sigma}(\pi_1(X)) \mid \forall i, \rho|_{I_{v_i}} \text{ has finite image}\}.$$

Let us also define categories consisting of isocrystals. Let \mathcal{G}_X be the category of finite etale Galois (connected) coverings of X and for an object $Y \rightarrow X$ in \mathcal{G}_X , let \overline{Y} be the normalization of \overline{X} in $k(Y)$, let \overline{Y}^{sm} be the smooth locus of \overline{Y} and put $G_Y := \text{Aut}(Y/X)$. Then G_Y acts on \overline{Y}^{sm} and so we can define the categories $\text{Isoc}([\overline{Y}^{\text{sm}}/G_Y])$ (resp. $F\text{-Isoc}([\overline{Y}^{\text{sm}}/G_Y])$, $F\text{-Isoc}([\overline{Y}^{\text{sm}}/G_Y])^\circ$) of convergent isocrystals (resp. convergent F -isocrystals, unit-root convergent F -isocrystals) on the quotient stack $[\overline{Y}^{\text{sm}}/G_Y]$ over K . Then we have the following:

Proposition 2.16. *Let $Y \rightarrow X, Y' \rightarrow X$ be objects in \mathcal{G}_X and let $f : Y' \rightarrow Y$ be a morphism in \mathcal{G}_X . Then we have the canonical functors*

$$\begin{aligned} f^* : F\text{-Isoc}([\overline{Y}^{\text{sm}}/G_Y]) &\longrightarrow F\text{-Isoc}([\overline{Y'}^{\text{sm}}/G_{Y'}]), \\ f^* : F\text{-Isoc}([\overline{Y}^{\text{sm}}/G_Y])^\circ &\longrightarrow F\text{-Isoc}([\overline{Y'}^{\text{sm}}/G_{Y'}])^\circ \end{aligned}$$

induced by f . When X is a curve, we also have the canonical functor

$$f^* : \text{Isoc}([\overline{Y}^{\text{sm}}/G_Y]) \longrightarrow \text{Isoc}([\overline{Y'}^{\text{sm}}/G_{Y'}]).$$

Proof. The latter assertion is obvious because we have the equalities $\overline{Y}^{\text{sm}} = \overline{Y}$, $\overline{Y'}^{\text{sm}} = \overline{Y'}$ in one dimensional case and we have a morphism $[\overline{Y'}/G_{Y'}] \rightarrow [\overline{Y}/G_Y]$. Let us prove the former assertion. Let us put $\overline{Y'}^{\text{sm}, \circ} := f^{-1}(\overline{Y}^{\text{sm}}) \cap \overline{Y'}^{\text{sm}}$. Then it is a $G_{Y'}$ -stable open subscheme of $\overline{Y'}^{\text{sm}}$ such that the complement $\overline{Y'}^{\text{sm}} \setminus \overline{Y'}^{\text{sm}, \circ}$ has codimension at least 2 in $\overline{Y'}^{\text{sm}}$. So, for $n = 0, 1, 2$, we have the equivalence of categories $F\text{-Isoc}(\overline{Y'}^{\text{sm}} \times G_{Y'}^n) \xrightarrow{\cong} F\text{-Isoc}(\overline{Y'}^{\text{sm}, \circ} \times G_{Y'}^n)$ by [47, 3.1] and this implies the equivalence $F\text{-Isoc}([\overline{Y'}^{\text{sm}}/G_{Y'}]) \xrightarrow{\cong} F\text{-Isoc}([\overline{Y'}^{\text{sm}, \circ}/G_{Y'}])$ (see Example 2.3). Then we can define the desired functor as the composite

$$F\text{-Isoc}([\overline{Y}^{\text{sm}}/G_Y]) \longrightarrow F\text{-Isoc}([\overline{Y'}^{\text{sm}, \circ}/G_{Y'}]) \xleftarrow{\cong} F\text{-Isoc}([\overline{Y'}^{\text{sm}}/G_{Y'}]),$$

where the first functor is induced by the morphism $[\overline{Y'}^{\text{sm}, \circ}/G_{Y'}] \rightarrow [\overline{Y}^{\text{sm}}/G_Y]$ and the second functor is the equivalence above. We can define the functor also in the unit-root case in the same way. \square

By Proposition 2.16, we can define the limit

$$\varinjlim_{Y \rightarrow X \in \mathcal{G}_X} F\text{-Isoc}([\overline{Y}^{\text{sm}}/G_Y]), \quad \varinjlim_{Y \rightarrow X \in \mathcal{G}_X} F\text{-Isoc}([\overline{Y}^{\text{sm}}/G_Y])^\circ$$

and we can define the limit $\varinjlim_{Y \rightarrow X \in \mathcal{G}_X} \text{Isoc}([\overline{Y}^{\text{sm}}/G_Y])$ when X is a curve. Now we prove the following theorem:

Theorem 2.17. *Let the notation be as above.*

(1) *There exists an equivalence of categories*

$$(2.5) \quad \text{Rep}_{K^\sigma}^{\text{fin}}(\pi_1(X)) \xrightarrow{\cong} F\text{-Isoc}^\dagger(X, \overline{X})^\circ$$

which is compatible with the equivalence (2.3) of Crew.

(2) *There exists an equivalence of categories*

$$(2.6) \quad \text{Rep}_{K^\sigma}^{\text{fin}}(\pi_1(X)) \xrightarrow{\cong} \varinjlim_{Y \rightarrow X \in \mathcal{G}_X} F\text{-Isoc}([\overline{Y}^{\text{sm}}/G_Y])^\circ$$

and a natural restriction functor

$$(2.7) \quad \varinjlim_{Y \rightarrow X \in \mathcal{G}_X} F\text{-Isoc}([\overline{Y}^{\text{sm}}/G_Y]) \longrightarrow F\text{-Isoc}^\dagger(X, \overline{X})$$

which makes the following diagram commutative:

$$(2.8) \quad \begin{array}{ccc} \mathrm{Rep}_{K^\sigma}^{\mathrm{fin}}(\pi_1(X)) & \xrightarrow{(2.6)} & \varinjlim_{Y \rightarrow X \in \mathcal{G}_X} F\text{-Isoc}([\overline{Y}^{\mathrm{sm}}/G_Y])^\circ \\ \parallel & & (2.7)^\circ \downarrow \\ \mathrm{Rep}_{K^\sigma}^{\mathrm{fin}}(\pi_1(X)) & \xrightarrow{(2.5)} & F\text{-Isoc}^\dagger(X, \overline{X})^\circ. \end{array}$$

When X is a curve, there exists also a natural restriction functor

$$(2.9) \quad \varinjlim_{Y \rightarrow X \in \mathcal{G}_X} \mathrm{Isoc}([\overline{Y}^{\mathrm{sm}}/G_Y]) \longrightarrow \mathrm{Isoc}^\dagger(X, \overline{X})$$

with $F\text{-(2.9)} = (2.7)$.

The equivalence (2.6) is the same as (0.7), which is a p -adic version of (0.2).

Proof. The equivalence (2.5) is already proven in [47, 4.2]. (In one dimensional case, this is a result of Tsuzuki [48]. In higher dimensional case, there is a related result by Kedlaya [23, 2.3.7, 2.3.9].)

Let us prove the equivalence (2.6). Let $\rho : \pi_1(X) \longrightarrow GL_d(K^\sigma)$ be an object in $\mathrm{Rep}_{K^\sigma}^{\mathrm{fin}}(\pi_1(X))$. By [47, 4.1, 4.2] or [23, 2.3.7, 2.3.9], there exists an object $\varphi : Y \longrightarrow X$ in \mathcal{G}_X such that, for any discrete valuation v of $k(Y)$ centered on $\overline{Y} \setminus Y$, the restriction of ρ to the inertia group I_v at v is trivial. In particular, $\rho|_{\pi_1(Y)}$ is unramified at any generic points of $\overline{Y}^{\mathrm{sm}} \setminus Y$ (note that $\overline{Y}^{\mathrm{sm}} \setminus Y$ is generically smooth). So, by Zariski-Nagata purity, we see that $\rho|_{\pi_1(Y)}$ factors through $\pi_1(\overline{Y}^{\mathrm{sm}})$. In this way, $\rho \in \mathrm{Rep}_{K^\sigma}^{\mathrm{fin}}(\pi_1(X)) \subseteq \mathrm{Rep}_{K^\sigma}(\pi_1(X)) \cong \mathrm{Sm}_{K^\sigma}(X)$ induces a G_Y -equivariant object in $\mathrm{Sm}_{K^\sigma}(\overline{Y}^{\mathrm{sm}})$ corresponding to $\rho|_{\pi_1(Y)}$.

On the other hand, let ρ be an object in $\mathrm{Rep}_{K^\sigma}(\pi_1(X))$ such that $\rho|_{\pi_1(Y)}$ factors through $\pi_1(\overline{Y}^{\mathrm{sm}})$ for some $Y \longrightarrow X$ in \mathcal{G}_X . Let \overline{X}' be the image of $\overline{Y}^{\mathrm{sm}}$ in \overline{X} . Then v_i 's are centered on \overline{X}' and so we can take an extension v'_i of v_i to $k(Y)$ whose center x is contained in $\overline{Y}^{\mathrm{sm}}$. Let $k(Y)_{v'_i}$ be the completion of $k(Y)$ with respect to v'_i and denote the valuation ring by $O_{v'_i}$. Then the composite

$$\mathrm{Spec} k(Y)_{v'_i} \longrightarrow \mathrm{Spec} k(Y) \longrightarrow Y \longrightarrow \overline{Y}^{\mathrm{sm}}$$

factors as

$$\mathrm{Spec} k(Y)_{v'_i} \longrightarrow \mathrm{Spec} O_{v'_i} \longrightarrow \mathrm{Spec} \mathcal{O}_{\overline{Y}^{\mathrm{sm}}, x} \longrightarrow \overline{Y}^{\mathrm{sm}}.$$

So we see that the restriction of $\rho|_{\pi_1(Y)} : \pi_1(Y) \longrightarrow \pi_1(\overline{Y}^{\mathrm{sm}}) \longrightarrow GL_d(K^\sigma)$ to $I_{v'_i}$ is trivial. Therefore, $\rho|_{I_{v'_i}}$ factors through the finite group $I_{v_i}/I_{v'_i}$ and we see that ρ is contained in $\mathrm{Rep}_{K^\sigma}^{\mathrm{fin}}(\pi_1(X))$.

Let $G_Y\text{-Sm}_{K^\sigma}(\overline{Y}^{\mathrm{sm}})$ be the category of G_Y -equivariant objects in the category $\mathrm{Sm}_{K^\sigma}(\overline{Y}^{\mathrm{sm}})$. Then we see from the argument in the previous two paragraphs that the above construction gives an equivalence

$$(2.10) \quad \mathrm{Rep}_{K^\sigma}^{\mathrm{fin}}(\pi_1(X)) \xrightarrow{=} \varinjlim_{Y \rightarrow X \in \mathcal{G}_X} G_Y\text{-Sm}_{K^\sigma}(\overline{Y}^{\mathrm{sm}}).$$

For $m = 0, 1, 2$, let $\overline{Y}_m^{\text{sm}}$ be the $(m+1)$ -fold fiber product of \overline{Y}^{sm} over $[\overline{Y}^{\text{sm}}/G_Y]$. Then they form a 2-truncated simplicial scheme $\overline{Y}_\bullet^{\text{sm}}$ and we have the canonical equivalence $G_Y\text{-Sm}_{K^\sigma}(\overline{Y}^{\text{sm}}) = \text{Sm}_{K^\sigma}(\overline{Y}_\bullet^{\text{sm}})$. Hence, by Crew's equivalence and Example 2.3, we have the equivalence of categories

$$\begin{aligned}
(2.11) \quad \varinjlim_{Y \rightarrow X \in \mathcal{G}_X} G_Y\text{-Sm}_{K^\sigma}(\overline{Y}^{\text{sm}}) &\xrightarrow{=} \varinjlim_{Y \rightarrow X \in \mathcal{G}_X} \text{Sm}_{K^\sigma}(\overline{Y}_\bullet^{\text{sm}}) \\
&\xrightarrow{=} \varinjlim_{Y \rightarrow X \in \mathcal{G}_X} F\text{-Isoc}(\overline{Y}_\bullet^{\text{sm}})^\circ \\
&\xrightarrow{=} \varinjlim_{Y \rightarrow X \in \mathcal{G}_X} F\text{-Isoc}([\overline{Y}^{\text{sm}}/G_Y])^\circ.
\end{aligned}$$

Combining this equivalence with (2.10), we obtain the equivalence (2.6).

Next let us define the functor (2.7). Let $\overline{Y}_\bullet^{\text{sm}}$ be as above, let Y_\bullet be the 2-truncated simplicial scheme such that Y_m ($m = 0, 1, 2$) is naturally the $(m+1)$ -fold fiber product of Y over X and let \overline{X}' be the image of \overline{Y}^{sm} in \overline{X} . We would like to define the composite functor

$$\begin{aligned}
(2.12) \quad F\text{-Isoc}([\overline{Y}^{\text{sm}}/G_Y]) &\xrightarrow{=} F\text{-Isoc}(\overline{Y}_\bullet^{\text{sm}}) \\
&\longrightarrow F\text{-Isoc}^\dagger(Y_\bullet, \overline{Y}_\bullet^{\text{sm}}) \\
&\xleftarrow{=} F\text{-Isoc}^\dagger(X, \overline{X}') \\
&\xleftarrow{=} F\text{-Isoc}^\dagger(X, \overline{X}).
\end{aligned}$$

In order that the functor is well-defined, we should prove that the third arrow in the above composite is an equivalence. (The fourth arrow is an equivalence by [47, 3.1].) Then it suffices to prove the equivalence

$$(2.13) \quad \text{Isoc}^\dagger(X, \overline{X}') \xrightarrow{=} \text{Isoc}^\dagger(Y_\bullet, \overline{Y}_\bullet^{\text{sm}}).$$

Note that the right hand side in (2.13) is the category of objects in $\text{Isoc}^\dagger(Y_0, \overline{Y}_0^{\text{sm}})$ endowed with equivariant G_Y -action. Then, if we denote the projection $\overline{Y}_0^{\text{sm}} \rightarrow \overline{X}'$ by π , we have the functor

$$(2.14) \quad \text{Isoc}^\dagger(Y_\bullet, \overline{Y}_\bullet^{\text{sm}}) \longrightarrow \text{Isoc}^\dagger(X, \overline{X}'); \quad \mathcal{E} \mapsto (\pi_* \mathcal{E})^{G_Y},$$

where π_* is the push-out functor defined by Tsuzuki [49]. By [49] and [21, 2.6.8], we have the trace morphisms $(\pi_* \pi^* \mathcal{E})^{G_Y} \rightarrow \mathcal{E}$, $\pi^*((\pi_* \mathcal{E})^{G_Y}) \rightarrow \mathcal{E}$, and they are isomorphic in $\text{Isoc}(X)$ and $\text{Isoc}(Y_\bullet)$ by etale descent. Since $\text{Isoc}^\dagger(X, \overline{X}') \rightarrow \text{Isoc}(X)$, $\text{Isoc}^\dagger(Y_\bullet, \overline{Y}_\bullet^{\text{sm}}) \rightarrow \text{Isoc}(Y_\bullet)$ are exact and faithful, they are actually isomorphic. Hence (2.14) is a quasi-inverse of (2.13) and so (2.13) is an equivalence. So the functor (2.12) is well-defined and it induces the functor (2.7). When X is a curve, we can define the functor (2.9) in the same way using (2.13). (Note that we do not have to use [47, 3.1] because $\overline{X} = \overline{X}'$ in the case of curves.)

To prove the commutativity of the diagram (2.8), it suffice to check it after we compose with the restriction functor

$$(2.15) \quad F\text{-Isoc}^\dagger(X, \overline{X})^\circ \longrightarrow F\text{-Isoc}(X)^\circ,$$

which is known to be fully faithful ([49]). Then $(2.15) \circ (2.12)^\circ$ is equal to the composite

$$\begin{aligned} F\text{-Isoc}([\overline{Y}^{\text{sm}}/G_Y])^\circ &\xrightarrow{=} F\text{-Isoc}(\overline{Y}_\bullet^{\text{sm}})^\circ \\ &\longrightarrow F\text{-Isoc}(Y_\bullet)^\circ \xleftarrow{=} F\text{-Isoc}(X)^\circ. \end{aligned}$$

Hence $(2.15) \circ (2.12)^\circ \circ (2.6)^\circ$ is the composite

$$\begin{aligned} \text{Rep}_{K^\sigma}^{\text{fin}}(\pi_1(X)) &\xrightarrow{(2.10)} \varinjlim_{Y \rightarrow X \in \mathcal{G}_X} G_Y\text{-Sm}_{K^\sigma}(\overline{Y}^{\text{sm}}) \\ &\xrightarrow{=} \varinjlim_{Y \rightarrow X \in \mathcal{G}_X} \text{Sm}_{K^\sigma}(\overline{Y}_\bullet^{\text{sm}}) \\ &\xrightarrow{G} \varinjlim_{Y \rightarrow X \in \mathcal{G}_X} F\text{-Isoc}(\overline{Y}_\bullet^{\text{sm}})^\circ \\ &\longrightarrow \varinjlim_{Y \rightarrow X \in \mathcal{G}_X} F\text{-Isoc}(Y_\bullet)^\circ \xleftarrow{=} F\text{-Isoc}(X)^\circ. \end{aligned}$$

By the functoriality of (2.4), it is equal to the composite

$$\begin{aligned} \text{Rep}_{K^\sigma}^{\text{fin}}(\pi_1(X)) &\xrightarrow{(2.10)} \varinjlim_{Y \rightarrow X \in \mathcal{G}_X} G_Y\text{-Sm}_{K^\sigma}(\overline{Y}^{\text{sm}}) \\ &\xrightarrow{=} \varinjlim_{Y \rightarrow X \in \mathcal{G}_X} \text{Sm}_{K^\sigma}(\overline{Y}_\bullet^{\text{sm}}) \\ &\longrightarrow \varinjlim_{Y \rightarrow X \in \mathcal{G}_X} \text{Sm}_{K^\sigma}(Y_\bullet) \\ &\xrightarrow{G} \varinjlim_{Y \rightarrow X \in \mathcal{G}_X} F\text{-Isoc}(Y_\bullet)^\circ \xleftarrow{=} F\text{-Isoc}(X)^\circ, \end{aligned}$$

and by definition of (2.10), it is rewritten as the composite

$$\begin{aligned} \text{Rep}_{K^\sigma}^{\text{fin}}(\pi_1(X)) &\xrightarrow{\subset} \text{Rep}_{K^\sigma}(\pi_1(X)) \\ &\xrightarrow{=} \text{Sm}_{K^\sigma}(X) \\ &\xrightarrow{=} \varinjlim_{Y \rightarrow X \in \mathcal{G}_X} \text{Sm}_{K^\sigma}(Y_\bullet) \\ &\xrightarrow{G} \varinjlim_{Y \rightarrow X \in \mathcal{G}_X} F\text{-Isoc}(Y_\bullet)^\circ \xleftarrow{=} F\text{-Isoc}(X)^\circ. \end{aligned}$$

Again by the functoriality of (2.4) and the assertion (1), it is further rewritten as the composite

$$\text{Rep}_{K^\sigma}^{\text{fin}}(\pi_1(X)) \xrightarrow{(2.5)} F\text{-Isoc}^\dagger(X, \overline{X})^\circ \xrightarrow{(2.15)} F\text{-Isoc}(X)^\circ.$$

So we have proved the commutativity of the diagram (2.8) and we are done. \square

Next let us consider ‘the tame variant’ of the equivalence (2.6). Let $\pi_1^t(X)$ be the tame fundamental group of X (tamely ramified at the valuations v_i ($1 \leq i \leq r$)) and let $\text{Rep}_{K^\sigma}(\pi_1^t(X))$ be the category of finite dimensional continuous representations of $\pi_1^t(X)$ over K^σ . On the other hand, let \mathcal{G}_X^t be the category of finite etale Galois tame covering of X (tamely ramified at v_i ($1 \leq i \leq r$)). For an object $Y \rightarrow X$ in \mathcal{G}_X^t , let $\overline{Y}, \overline{Y}^{\text{sm}}, G_Y$ be as before. Then we have the following:

Theorem 2.18. *Let the notations be as above. Then there exists an equivalence of categories*

$$(2.16) \quad \text{Rep}_{K^\sigma}(\pi_1^t(X)) \xrightarrow{\cong} \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} F\text{-Isoc}([\overline{Y}^{\text{sm}}/G_Y])^\circ$$

and a natural restriction functor

$$(2.17) \quad \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} F\text{-Isoc}([\overline{Y}^{\text{sm}}/G_Y]) \longrightarrow F\text{-Isoc}^\dagger(X, \overline{X})$$

which makes the following diagram commutative:

$$(2.18) \quad \begin{array}{ccc} \text{Rep}_{K^\sigma}(\pi_1^t(X)) & \xrightarrow{(2.16)} & \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} F\text{-Isoc}([\overline{Y}^{\text{sm}}/G_Y])^\circ \\ \cap \downarrow & & (2.17)^\circ \downarrow \\ \text{Rep}_{K^\sigma}^{\text{fin}}(\pi_1(X)) & \xrightarrow{(2.5)} & F\text{-Isoc}^\dagger(X, \overline{X})^\circ. \end{array}$$

When X is a curve, we have also a natural restriction functor

$$(2.19) \quad \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} \text{Isoc}([\overline{Y}^{\text{sm}}/G_Y]) \longrightarrow \text{Isoc}^\dagger(X, \overline{X})$$

with $F\text{-}(2.19) = (2.17)$.

The equivalence (2.16) is the same as (0.8), which is a tame p -adic version of (0.2). Before the proof of Theorem 2.18, we prove a lemma.

Lemma 2.19. *Let $X \hookrightarrow \overline{X}$, v_i ($1 \leq i \leq r$) be as above and let us take $\rho \in \text{Rep}_{K^\sigma}(\pi_1^t(X))$. Let v be a discrete valuation on $k(X)$ centered on \overline{X} , let $k(X)_v$ be the completion of $k(X)$ with respect to v and let I_v be the inertia group of $k(X)_v$. Then $|\text{Im}(\rho|_{I_v})|$ is finite and prime to p . (In particular, we have the inclusion $\text{Rep}_{K^\sigma}(\pi_1^t(X)) \subseteq \text{Rep}_{K^\sigma}^{\text{fin}}(\pi_1(X))$.)*

Proof. Let us take a suitable O_K^σ -lattice $\pi_1^t(X) \longrightarrow GL_d(O_K^\sigma)$ of ρ . First we prove the lemma in the case where $v = v_i$ for some i . In this case, $\rho|_{I_v}$ factors through the tame quotient I_v^t by definition. Note that I_v^t is a pro-prime-to- p group and that $N := \text{Ker}(GL_d(O_K^\sigma) \longrightarrow GL_d(O_K^\sigma/2pO_K^\sigma))$ is a pro- p group. So the image of $\rho|_{I_v} : I_v \longrightarrow I_v^t \longrightarrow GL_d(O_K^\sigma)$ is isomorphic to the image of the composite

$I_v \longrightarrow I_v^t \longrightarrow GL_d(O_K^\sigma) \longrightarrow GL_d(O_K^\sigma)/N$. So $|\text{Im}(\rho|_{I_v})|$ is finite. Then, since I_v^t is a pro-prime-to- p group, we see also that $|\text{Im}(\rho|_{I_v})|$ is prime to p . So we are done in this case. Note also that, since we have the isomorphism $I_v^t \cong \prod_{l \neq p} \mathbb{Z}_l$ and $\text{Ker}(I_v \longrightarrow I_v^t)$ is a pro- p -group, $\rho|_{I_v}$ factors through I_v/J for any $J \triangleleft I_v$ such that $|I_v/J|$ is prime to p and divisible by $|\text{Im}(\rho|_{I_v})|$.

Next we prove the lemma for general v . Let x be the center of v and take an open neighborhood $\overline{U} \subseteq \overline{X}$ of x such that \overline{U} admits a smooth morphism $f : \overline{U} \longrightarrow \mathbb{A}_k^r = \text{Spec } k[t_1, \dots, t_r]$ with $f^{-1}(\{t_i = 0\}) = \overline{U} \cap Z_i$ ($1 \leq i \leq r$). Let us put $U := \overline{U} \cap X = f^{-1}(\mathbb{G}_{m,k}^r)$. Let us take a positive integer n prime to p which is divisible by all the $|\text{Im}(\rho|_{I_{v_i}})|$'s for $1 \leq i \leq r$ (there exists such n by the lemma for v_i 's). Let $n : \mathbb{A}_k^r \longrightarrow \mathbb{A}_k^r$ be the n -th power map and put $\overline{U}' := \overline{U} \times_{\mathbb{A}_k^r, n} \mathbb{A}_k^r$, $U' := U \times_{\mathbb{A}_k^r, n} \mathbb{A}_k^r$. Let us denote the inverse image of the i -th coordinate hyperplane by the projection $\overline{U}' = \overline{U} \times_{\mathbb{A}_k^r, n} \mathbb{A}_k^r \longrightarrow \mathbb{A}_k^r$ by Z'_i ($1 \leq i \leq r$). Then we have $\overline{U}' \setminus U' =: Z' = \bigcup_{i=1}^r Z'_i$ and it is a simple normal crossing divisor. For i with $Z'_i \neq \emptyset$, let v'_i be the discrete valuation on $k(U')$ corresponding to the generic point of Z'_i , let $k(U')_{v'_i}$ be the completion of $k(U')$ with respect to v'_i and let $I_{v'_i}$ be the inertia group of $k(U')_{v'_i}$. Then, by definition, v'_i is an extension of v_i and we have $|I_{v_i}/I_{v'_i}| = n$. So, by the argument in the previous paragraph, $\rho|_{I_{v_i}}$ factors through $I_{v_i}/I_{v'_i}$. So $\rho|_{I_{v'_i}}$ is trivial and hence $\rho|_{\pi_1(U')}$ factors through $\pi_1(\overline{U}')$. Now let v' be an extension of the valuation v to $k(U')$ centered on \overline{U}' and let $x' \in \overline{U}'$ be the center of v' . Let $k(U')_{v'}$ be the completion of $k(U')$ with respect to v' and denote the valuation ring by $O_{v'}$. Then the composite

$$\text{Spec } k(U')_{v'} \longrightarrow \text{Spec } k(U') \longrightarrow U' \longrightarrow \overline{U}'$$

factors as

$$\text{Spec } k(U')_{v'} \longrightarrow \text{Spec } O_{v'} \longrightarrow \text{Spec } \mathcal{O}_{\overline{U}', x'} \longrightarrow \overline{U}'.$$

So we see that the restriction of $\rho|_{\pi_1(U')} : \pi_1(U') \longrightarrow \pi_1(\overline{U}') \longrightarrow GL_d(\mathcal{O}_K^\sigma)$ to $I_{v'}$ is trivial. Hence $\rho|_{I_v}$ factors through $I_v/I_{v'}$. Since $|I_v/I_{v'}|$ divides $[k(U') : k(U)] = n^r$, we can conclude that $|\text{Im}(\rho|_{I_v})|$ is finite and prime to p . So we are done. \square

Now we prove Theorem 2.18, using the above lemma.

Proof of Theorem 2.18. First let us prove the equivalence (2.16). Let ρ be an object in $\text{Rep}_{K^\sigma}(\pi_1^t(X))$ and take a suitable O_K^σ -lattice $\pi_1^t(X) \longrightarrow GL_d(O_K^\sigma)$ of ρ . Let $Y \longrightarrow X$ be a finite etale Galois tame covering of X (tamely ramified along v_i 's) which corresponds to the subgroup $\text{Ker}(\pi_1^t(X) \xrightarrow{\rho} GL_d(O_K^\sigma) \twoheadrightarrow GL_d(O_K^\sigma/2pO_K^\sigma))$. Let v be any discrete valuation of $k(Y)$ centered on \overline{Y} . Then $v|_{k(X)}$ is a discrete valuation of $k(X)$ centered on \overline{X} . So, by Lemma 2.19, $|\text{Im}(\rho|_{I_{v|_{k(X)}}})|$ is finite and prime to p . Hence so is $|\text{Im}(\rho|_{I_v})|$. On the other hand, $\text{Im}(\rho|_{I_v})$ is contained in $\text{Ker}(GL_d(O_K^\sigma) \twoheadrightarrow GL_d(O_K^\sigma/2pO_K^\sigma))$, which is a pro- p group. Hence $\rho|_{I_v}$ is trivial.

By using this fact, we see in the same way as the proof of Theorem 2.17 that $\rho|_{\pi_1(Y)}$ factors through $\pi_1(\overline{Y}^{\text{sm}})$ and so induces a G_Y -equivariant object of $\text{Sm}_{K^\sigma}(\overline{Y}^{\text{sm}})$.

On the other hand, let ρ be an object in $\text{Rep}_{K^\sigma}(\pi_1(X))$ such that $\rho|_{\pi_1(Y)}$ factors through $\pi_1(\overline{Y}^{\text{sm}})$ for some $Y \rightarrow X$ in \mathcal{G}_X^t . Then we see in the same way as the proof of Theorem 2.17 that, for an extension v'_i of v_i to $k(Y)$ ($1 \leq i \leq r$) centered on \overline{Y}^{sm} , $\rho|_{I_{v'_i}}$ is trivial. Then $\rho|_{I_{v_i}}$ factors through $I_{v_i}/I_{v'_i}$ and this is a quotient of the tame inertia quotient $I_{v_i}^t$ of I_{v_i} because $Y \rightarrow X$ is in \mathcal{G}_X^t . Hence ρ factors through $\pi_1^t(X)$. So we have an equivalence

$$(2.20) \quad \text{Rep}_{K^\sigma}^{\text{fin}}(\pi_1^t(X)) \xrightarrow{=} \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} G_Y\text{-Sm}_{K^\sigma}(\overline{Y}^{\text{sm}}).$$

(Here $G_Y\text{-Sm}_{K^\sigma}(\overline{Y}^{\text{sm}})$ is as in the proof of Theorem 2.17.) Combining this with the equivalence

$$\varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} G_Y\text{-Sm}_{K^\sigma}(\overline{Y}^{\text{sm}}) \xrightarrow{=} \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} F\text{-Isoc}([\overline{Y}^{\text{sm}}/G_Y])^\circ$$

which is defined in the same way as (2.11), we obtain the equivalence (2.16).

The functor (2.17) is defined as the composite of the canonical ‘inclusion functor’ $\varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} F\text{-Isoc}([\overline{Y}^{\text{sm}}/G_Y]) \rightarrow \varinjlim_{Y \rightarrow X \in \mathcal{G}_X} F\text{-Isoc}([\overline{Y}^{\text{sm}}/G_Y])$ and the functor (2.7). By construction, we have the commutative diagram

$$\begin{array}{ccc} \text{Rep}(\pi_1^t(X)) & \xrightarrow{(2.16)} & \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} F\text{-Isoc}([\overline{Y}^{\text{sm}}/G_Y])^\circ \\ \cap \downarrow & & \text{incl.} \downarrow \\ \text{Rep}^{\text{fin}}(\pi_1(X)) & \xrightarrow{(2.6)} & \varinjlim_{Y \rightarrow X \in \mathcal{G}_X} F\text{-Isoc}([\overline{Y}^{\text{sm}}/G_Y])^\circ \end{array}$$

(where incl. denotes the ‘inclusion functor’). By combining this with (2.7), we obtain the commutative diagram (2.18). When X is a curve, we define the functor (2.19) as the composite of the canonical ‘inclusion functor’ $\varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} \text{Isoc}([\overline{Y}^{\text{sm}}/G_Y]) \rightarrow \varinjlim_{Y \rightarrow X \in \mathcal{G}_X} \text{Isoc}([\overline{Y}^{\text{sm}}/G_Y])$ and the functor (2.9). Then it is easy to see the equality $F\text{-(2.19)} = (2.17)$. So we are done. \square

2.3 Stack of roots

In this subsection, we recall the notion of stack of roots, which is treated in [5], [3], [4], [14]. We also define the canonical log structure on it and we also introduce a ‘bisimplicial resolution’ of it which we use later.

For $r \in \mathbb{N}$, let $[\mathbb{A}_k^r/\mathbb{G}_{m,k}^r]$ be the stack over k which is the quotient of \mathbb{A}_k^r by the canonical action of $\mathbb{G}_{m,k}^r$. It is known [34, 5.13] that it classifies pairs (M, γ) , where M is a fine log structure and $\gamma : \mathbb{N}^r \rightarrow \overline{M} := M/\mathcal{O}^\times$ is a homomorphism

of monoid sheaves which lifts to a chart etale locally. By [34, 5.14, 5.15], there exists the canonical fine log structure $M_{[\mathbb{A}_k^r/\mathbb{G}_{m,k}^r]}$ on $[\mathbb{A}_k^r/\mathbb{G}_{m,k}^r]$ endowed with $\mathbb{N}^r \longrightarrow \overline{M_{[\mathbb{A}_k^r/\mathbb{G}_{m,k}^r]}} (:= M_{[\mathbb{A}_k^r/\mathbb{G}_{m,k}^r]}/\mathcal{O}_{[\mathbb{A}_k^r/\mathbb{G}_{m,k}^r]}^\times)$ such that the above (M, γ) is realized as the pull-back of $M_{[\mathbb{A}_k^r/\mathbb{G}_{m,k}^r]}$.

Now let $X \hookrightarrow \overline{X}$ be an open immersion of smooth varieties over k such that $\overline{X} \setminus X =: Z = \bigcup_{i=1}^r Z_i$ is a simple normal crossing divisor (each Z_i being irreducible). Denote the fine log structure on \overline{X} associated to Z by $M_{\overline{X}}$. Then we have the morphism $\overline{X} \longrightarrow [\mathbb{A}_k^r/\mathbb{G}_{m,k}^r]$ defined by $(M_{\overline{X}}, \gamma)$, where γ is the homomorphism of monoid sheaves

$$\mathbb{N}^r \longrightarrow \overline{M_{\overline{X}}} = \bigoplus_{i=1}^r \mathbb{N}_{Z_i}$$

(where \mathbb{N}_{Z_i} is the direct image to \overline{X} of the constant sheaf on Z_i with fiber \mathbb{N}) induced by the maps $\mathbb{N} \longrightarrow \mathbb{N}_{Z_i}$ ($1 \leq i \leq r$) which are adjoint to the identity. For $n \in \mathbb{N}$ prime to p , let $n : [\mathbb{A}_k^r/\mathbb{G}_{m,k}^r] \longrightarrow [\mathbb{A}_k^r/\mathbb{G}_{m,k}^r]$ be the morphism induced by the n -th power map. Then we define

$$(\overline{X}, Z)^{1/n} := \overline{X} \times_{[\mathbb{A}_k^r/\mathbb{G}_{m,k}^r], n} [\mathbb{A}_k^r/\mathbb{G}_{m,k}^r]$$

and call it the stack of $(n$ -th) roots of (\overline{X}, Z) . (Note that we always assume that n is prime to p in this paper.)

It has the following local description ([3], [4], see also [16, complement 1]): Assume $\overline{X} = \text{Spec } R$ is affine and assume that each Z_i ($1 \leq i \leq r$) is equal to the zero locus of some element $t_i \in R$. Let us put $R' := R[s_1, \dots, s_r]/(s_1^n - t_1, \dots, s_r^n - t_r)$, $\overline{X}' := \text{Spec } R'$, let Z'_i ($1 \leq i \leq r$) be the divisor of \overline{X}' defined by s_i and let $M_{\overline{X}'}$ be the log structure associated to the simple normal crossing divisor $\bigcup_{i=1}^r Z'_i$. Then we have the diagram

$$(2.21) \quad \begin{array}{ccccc} \mathbb{N}^r & \xrightarrow{\gamma} & \overline{M_{\overline{X}}} & \equiv & \bigoplus_{i=1}^r \mathbb{N}_{Z_i} \\ n \downarrow & & \downarrow & & n \downarrow \\ \mathbb{N}^r & \xrightarrow{\gamma} & \overline{M_{\overline{X}'}} & \equiv & \bigoplus_{i=1}^r \mathbb{N}_{Z'_i} \end{array}$$

(where γ 's are defined in the same way as before and n denotes the multiplication-by- n maps) and it induces the morphism

$$(2.22) \quad \overline{X}' \longrightarrow (\overline{X}, Z)^{1/n}.$$

Moreover, \overline{X}' admits the canonical action of μ_n^r ($:=$ the product of r copies of the group scheme of n -th roots of unity) defined as the action on s_i 's. Since the lower horizontal line of (2.21) is invariant by the action of μ_n^r , we see that the morphism (2.22) is stable under the action of μ_n^r and induces the morphism

$$(2.23) \quad [\overline{X}'/\mu_n^r] \xrightarrow{=} (\overline{X}, Z)^{1/n}$$

which is known to be an isomorphism ([3], [4]). So, if we define $\overline{X}'_m (m = 0, 1, 2)$ as the $(m + 1)$ -fold fiber product of \overline{X}' over $[\overline{X}'/\mu_n^r] \cong (\overline{X}, Z)^{1/n}$, \overline{X}'_\bullet forms a 2-truncated simplicial scheme over $(\overline{X}, Z)^{1/n}$ and we have the equivalences

$$(2.24) \quad \begin{aligned} \mathrm{Isoc}((\overline{X}, Z)^{1/n}) &\xrightarrow{\cong} \mathrm{Isoc}(\overline{X}'_\bullet), \quad F\text{-}\mathrm{Isoc}((\overline{X}, Z)^{1/n}) \xrightarrow{\cong} F\text{-}\mathrm{Isoc}(\overline{X}'_\bullet), \\ F\text{-}\mathrm{Isoc}((\overline{X}, Z)^{1/n})^\circ &\xrightarrow{\cong} F\text{-}\mathrm{Isoc}(\overline{X}'_\bullet)^\circ. \end{aligned}$$

We need a globalized version of (the log version of) the equivalences (2.24). To describe it, we introduce several new notions.

Let $X \hookrightarrow \overline{X}$, $\overline{X} \setminus X =: Z = \bigcup_{i=1}^r Z_i$, $M_{\overline{X}}$ be as above and fix a positive integer n prime to p . In this subsection, a chart for (\overline{X}, Z) is defined to be a pair $(\overline{Y}, \{t_i\}_{i=1}^r)$ consisting of an affine scheme \overline{Y} endowed with a surjective etale morphism $\overline{Y} \rightarrow \overline{X}$ and sections $t_i \in \Gamma(\overline{Y}, \mathcal{O}_{\overline{Y}})$ ($1 \leq i \leq r$) with $\overline{Y} \times_{\overline{X}} Z_i = \{t_i = 0\}$. (This notion of a chart is not so different from that of a chart of $M_{\overline{X}}$.) As a variant of it, we define the notion of a multichart for (\overline{X}, Z) to be a triple $(\overline{Y}, J, \{t_{ij}\}_{1 \leq i \leq r, j \in J})$ consisting of an affine scheme \overline{Y} endowed with a surjective etale morphism $\overline{Y} \rightarrow \overline{X}$, a non-empty finite set J and sections $t_{ij} \in \Gamma(\overline{Y}, \mathcal{O}_{\overline{Y}})$ ($1 \leq i \leq r, j \in J$) with $\overline{Y} \times_{\overline{X}} Z_i = \{t_{ij} = 0\}$ for any $j \in J$. A morphism $(\overline{Y}, J, \{t_{ij}\}) \rightarrow (\overline{Y}', J', \{t'_{ij}\})$ of multicharts for (\overline{X}, Z) is defined to be a pair $(\varphi, \varphi^\#)$ consisting of $\varphi : \overline{Y} \rightarrow \overline{Y}'$ and $\varphi^\# : J' \rightarrow J$ with $\varphi^* t'_{ij} = t_{i\varphi^\#(j)}$ ($1 \leq i \leq r, j \in J'$).

Now let us take a multichart $(\overline{Y}, J, \{t_{ij}\})$ and put $\overline{Y} =: \mathrm{Spec} R$. Let us denote $M_{\overline{X}}|_{\overline{Y}}$ simply by $M_{\overline{Y}}$. For $j \in J$, let us put

$$R'_j := R[s_{ij}]_{1 \leq i \leq r} / (s_{ij}^n - t_{ij}), \quad \overline{Y}'_j := \mathrm{Spec} R'_j.$$

Let $M_{\overline{Y}'_j}$ be the log structure on \overline{Y}'_j associated to the homomorphism $\alpha'_j : \mathbb{N}^r \rightarrow R'_j; e_i \mapsto s_{ij}$. Let $\varphi_j : (\overline{Y}'_j, M_{\overline{Y}'_j}) \rightarrow (\overline{Y}, M_{\overline{Y}})$ be the morphism associated to the diagram

$$\begin{array}{ccc} R'_j & \longleftarrow & R \\ \alpha'_j \uparrow & & \alpha_j^\circ \uparrow \\ \mathbb{N}^r & \xleftarrow{n} & \mathbb{N}^r, \end{array}$$

where the upper horizontal arrow is the natural inclusion, $\alpha_j^\circ : \mathbb{N}^r \rightarrow R$ is the map sending e_i to t_{ij} and $n : \mathbb{N}^r \rightarrow \mathbb{N}^r$ is the multiplication by n . Then φ_j is a finite Kummer log etale morphism. We have the natural action of μ_n^r (action on s_{ij} 's) on $(\overline{Y}'_j, M_{\overline{Y}'_j})$ such that φ_j is μ_n^r -stable.

Now let $(\overline{Y}_0, M_{\overline{Y}_0})$ be the fiber product of $(\overline{Y}'_j, M_{\overline{Y}'_j})$'s ($j \in J$) over $(\overline{Y}, M_{\overline{Y}})$ in the category of fs log schemes and for $m = 0, 1, 2$, let $(\overline{Y}_m, M_{\overline{Y}_m})$ be the $(m + 1)$ -fold fiber product of $(\overline{Y}_0, M_{\overline{Y}_0})$ over $(\overline{Y}, M_{\overline{Y}})$ in the category of fs log schemes. If we put $c := |J|$, $(\overline{Y}_m, M_{\overline{Y}_m})$ admits a natural action of $\mu_n^{rc(m+1)}$. Then we have the following properties:

Proposition 2.20. *Let the notations be as above. Then:*

- (1) *There exists an isomorphism $\overline{Y}_m \cong \overline{Y}_0 \times \mu_n^{rcm}$ such that the morphism $\overline{Y}_0 \times \mu_n^{rc} \cong \overline{Y}_\bullet \longrightarrow [\overline{Y}_0/\mu_n^{rc}]$ is the 2-truncated étale Čech hypercovering associated to the quotient map $\overline{Y}_0 \longrightarrow [\overline{Y}_0/\mu_n^{rc}]$.*
- (2) *There exists an isomorphism*

$$(2.25) \quad [\overline{Y}_0/\mu_n^{rc}] \xrightarrow{=} \overline{Y} \times_{\overline{X}} (\overline{X}, Z)^{1/n}$$

such that the log structure associated to the composite

$$\overline{Y}_m \xrightarrow{\text{any proj.}} \overline{Y}_0 \longrightarrow [\overline{Y}_0/\mu_n^{rc}] \xrightarrow{(2.25)} \overline{Y} \times_{\overline{X}} (\overline{X}, Z)^{1/n} \longrightarrow [\mathbb{A}_k^r/\mathbb{G}_{m,k}^r]$$

via [34, 5.13] is equal to $M_{\overline{Y}_m}$.

Proof. In this proof, we put $J := \{1, 2, \dots, c\}$. Let us put $u_{ij} := t_{ij}t_{i1}^{-1} \in R^\times$ ($1 \leq i \leq r, j \in J$) and put

$$R_0 := R[s_{i1} \ (1 \leq i \leq r), s'_{ij} \ (1 \leq i \leq r, 2 \leq j \leq c)] / (s_{i1}^n - t_{i1} \ (1 \leq i \leq r), s'_{ij}{}^n - u_{ij} \ (1 \leq i \leq r, 2 \leq j \leq c)).$$

Let us define the log structure N_0 on $\text{Spec } R_0$ as the one associated to the homomorphism $\alpha_1 : \mathbb{N}^r \longrightarrow R_0; e_i \mapsto s_{i1}$. Note that, since s'_{ij} 's are invertible, N_0 is associated also to the homomorphism $\alpha_j : \mathbb{N}^r \longrightarrow R_0; e_i \mapsto s_{i1}s'_{ij}$ for $2 \leq j \leq c$. Let $\varphi : (\text{Spec } R_0, N_0) \longrightarrow (\overline{Y}, M_{\overline{Y}})$ be the morphism associated to the diagram

$$\begin{array}{ccc} R_0 & \longleftarrow & R \\ \alpha_1 \uparrow & & \alpha_1^\circ \uparrow \\ \mathbb{N}^r & \xleftarrow{n} & \mathbb{N}^r, \end{array}$$

where the upper horizontal arrow is the natural inclusion, and let $\psi'_j : (\text{Spec } R_0, N_0) \longrightarrow (\overline{Y}'_j, M_{\overline{Y}'_j})$ be the morphism associated to the diagram

$$\begin{array}{ccc} R_0 & \longleftarrow & R'_j \\ \alpha_j \uparrow & & \alpha'_j \uparrow \\ \mathbb{N}^r & \xleftarrow{=} & \mathbb{N}^r, \end{array}$$

where the upper horizontal arrow is the homomorphism over R with $s_{i1} \mapsto s_{s1}$ when $j = 1$, $s_{ij} \mapsto s_{i1}s'_{ij}$ when $2 \leq j \leq c$. Then we have $\varphi_j \circ \psi'_j = \varphi$ for all $j \in J$. So ψ'_j 's induce the morphism $\psi_0 : (\text{Spec } R_0, N_0) \longrightarrow (\overline{Y}_0, M_{\overline{Y}_0})$ over $(\overline{Y}, M_{\overline{Y}})$.

Let us prove that ψ_0 is an isomorphism. To do so, we can work etale locally on \overline{Y} and so we may assume that R contains $u_{ij}^{1/n}$ for $1 \leq i \leq r, 2 \leq j \leq c$. Also, we put $u_{i1}^{1/n} = 1$. In this situation, φ_j is associated also to the diagram

$$\begin{array}{ccc} R'_j & \longleftarrow & R \\ \alpha''_j \uparrow & & \alpha_1^\circ \uparrow \\ \mathbb{N}^r & \xleftarrow{n} & \mathbb{N}^r, \end{array}$$

where α''_j is the homomorphism $\mathbb{N}^r \longrightarrow R'_j$; $e_i \mapsto s_{ij}u_{ij}^{1/n}$. Let us define Q to be the push-out of homomorphisms of monoids

$$\mathbb{N}^r \xrightarrow{n \circ (j\text{-th incl.})} \bigoplus_{1 \leq j \leq c} \mathbb{N}^r = \bigoplus_{\substack{1 \leq i \leq r \\ 1 \leq j \leq c}} \mathbb{N} \quad (1 \leq j \leq c)$$

and let Q^{sat} be the saturation of Q . Then we have the isomorphism

$$(2.26) \quad \mathbb{N}^r \oplus \left(\bigoplus_{\substack{1 \leq i \leq r \\ 2 \leq j \leq c}} \mathbb{Z}/n\mathbb{Z} \right) \xrightarrow{=} Q^{\text{sat}}$$

characterized by $(e_i, 0) \mapsto e_{i1}, (e_i, -e_{ij}) \mapsto e_{ij}$. Let $\alpha'' : \mathbb{Z}[Q] \longrightarrow \bigotimes_{j=1}^c R'_j$ be the homomorphism induced by α''_j 's. Then we can calculate $\Gamma(\overline{Y}_0, \mathcal{O}_{\overline{Y}_0})$ by the equalities

$$\begin{aligned} & \Gamma(\overline{Y}_0, \mathcal{O}_{\overline{Y}_0}) \\ &= \left(\bigotimes_{j=1}^c R'_j \right) \otimes_{\alpha'', \mathbb{Z}[Q]} \mathbb{Z}[Q^{\text{sat}}] \\ &= (R[s_{ij}]_{1 \leq i \leq r, 1 \leq j \leq c} / (s_{ij}^n - t_{ij})) \otimes_{\alpha'', \mathbb{Z}[Q]} \mathbb{Z}[Q^{\text{sat}}] \\ &= (R[s_{ij}]_{1 \leq i \leq r, 1 \leq j \leq c} / ((s_{ij}u_{ij}^{1/n})^n - t_{i1})) \otimes_{\alpha'', \mathbb{Z}[Q]} \mathbb{Z}[Q^{\text{sat}}] \\ &= R[\{s_{i1}\}_{1 \leq i \leq r}, \{s''_{ij}\}_{1 \leq i \leq r, 2 \leq j \leq c}] / (s_{i1}^n - t_{i1}, (s''_{ij})^n - 1) \quad (s''_{ij} = s_{i1}(s_{ij}u_{ij}^{1/n})^{-1}) \\ &= R[\{s_{i1}\}_{1 \leq i \leq r}, \{s'_{ij}\}_{1 \leq i \leq r, 2 \leq j \leq c}] / (s_{i1}^n - t_{i1}, (s'_{ij})^n - u_{ij}) \quad (s'_{ij} = u_{ij}^{1/n} s''_{ij}) \\ &= R_0 \end{aligned}$$

which is equal to the ring homomorphism induced by ψ_0 . Moreover, it is easy to see that $\psi_0^* M_{\overline{Y}_0}$, being equal to the log structure associated to $Q^{\text{sat}} \longrightarrow R_0$ induced by the above diagram, is equal to the log structure associated to $\mathbb{N}^r \hookrightarrow Q^{\text{sat}} \longrightarrow R_0$, that is, the log structure N_0 . So ψ_0 is an isomorphism. Note also that we have the natural action of $\mu_n^r \times \mu_n^{r(c-1)}$ (action on s_{i1} 's and s'_{ij} 's) on $(\text{Spec } R_0, N_0)$. Then, by definition, we can see that the isomorphism ψ_0 is equivariant with the group isomorphism

$$\mu_n^r \times \mu_n^{r(c-1)} \xrightarrow{=} \mu_n^{rc}$$

defined by $(\zeta, 1) \mapsto (\zeta, \dots, \zeta) (\zeta \in \mu_n^r)$, $(1, \eta) \mapsto (1, \eta) (\eta \in \mu_n^{r(c-1)})$.

Let us put $Z_{0,i} := \psi_0(\{s_{i1} = 0\}) \subseteq \overline{Y}_0$. Then we see from the isomorphism ψ_0 that the log structure $M_{\overline{Y}_0}$ is associated to $\bigcup_{i=1}^r Z_{0,i}$. Then the commutative diagram

$$(2.27) \quad \begin{array}{ccc} \mathbb{N}^r & \longrightarrow & \overline{M_{\overline{Y}}} = \bigoplus_{i=1}^r \mathbb{N}_{\overline{Y} \times_{\overline{X}} Z_i} \\ n \downarrow & & n \downarrow \\ \mathbb{N}^r & \longrightarrow & \overline{M_{\overline{Y}_0}} = \bigoplus_{i=1}^r \mathbb{N}_{Z_{0,i}} \end{array}$$

induces the morphism $\overline{Y}_0 \longrightarrow \overline{Y} \times_{\overline{X}} (\overline{X}, Z)^{1/n}$, and since the lower horizontal line of (2.27) is invariant under the action of μ_n^{rc} , this diagram further induces the morphism $[\overline{Y}_0/\mu_n^{rc}] \longrightarrow \overline{Y} \times_{\overline{X}} (\overline{X}, Z)^{1/n}$, which is the definition of the morphism (2.25).

Let us prove that this is an isomorphism. If we start the constuction of (2.25) from the multichart $(\overline{Y}, \{1\}, \{t_{i1}\})$ instead of $(\overline{Y}, J, \{t_{ij}\})$, we obtain the morphism

$$(2.28) \quad [\overline{Y}'_1/\mu_n^r] \longrightarrow \overline{Y} \times_{\overline{X}} (\overline{X}, Z)^{1/n},$$

and this is an isomorphism because the construction of it is the same as that of the isomorphism (2.23). Moreover, the morphism of multichart $\iota : (\overline{Y}, J, \{t_{ij}\}) \longrightarrow (\overline{Y}, \{1\}, \{t_{i1}\})$ defined by the identity $\overline{Y} \xrightarrow{=} \overline{Y}$ and the inclusion $\{1\} \hookrightarrow J$ induces the factorization

$$[\overline{Y}_0/\mu_n^{rc}] \xrightarrow{\iota'} [\overline{Y}'_1/\mu_n^r] \xrightarrow{(2.28)} \overline{Y} \times_{\overline{X}} (\overline{X}, Z)^{1/n}$$

of (2.25), where ι' is the map induced by ι . So it suffices to prove that ι' above is an isomorphism, and one can see it because ι' is factorized to the following sequence of isomorphisms:

$$\begin{aligned} & [\overline{Y}_0/\mu_n^{rc}] \\ & \xrightarrow{\psi_0^{-1}} [\mathrm{Spec} R_0/(\mu_n^r \times \mu_n^{r(c-1)})] \\ & \xrightarrow{=} [\mathrm{Spec} R'_1/\mu_n^r] \times_{\mathrm{Spec} R} [(\mathrm{Spec} R[s'_{ij}]_{1 \leq i \leq r, 2 \leq j \leq c} / (s'_{ij}{}^n - u_{ij})_{1 \leq i \leq r, 2 \leq j \leq c}) / \mu_n^{r(c-1)}] \\ & \xrightarrow{\mathrm{proj}} [\mathrm{Spec} R'_1/\mu_n^r] = [\overline{Y}'_1/\mu_n^r]. \end{aligned}$$

So we have shown that (2.25) is an isomorphism.

Next we prove the assertion (1). For $m = 1, 2$, we put

$$R_m := R_0[s''_{ijl}]_{1 \leq i \leq r, 1 \leq j \leq c, 1 \leq l \leq m} / (s''_{ijl}{}^n - 1)$$

and let N_m be the log structure on $\mathrm{Spec} R_m$ associated to the homomorphism $\mathbb{N}^r \xrightarrow{\alpha_1} R_0 \xrightarrow{\subset} R_m$. Let $\pi_0 : (\mathrm{Spec} R_1, N_1) \longrightarrow (\mathrm{Spec} R_0, N_0)$ be the morphism over $(\overline{Y}, M_{\overline{Y}})$ defined by the natural inclusion $R_0 \xrightarrow{\subset} R_1$, and let $\pi_1 : (\mathrm{Spec} R_1, N_1) \longrightarrow (\mathrm{Spec} R_0, N_0)$ be the morphism over $(\overline{Y}, M_{\overline{Y}})$ defined by

$$R_0 \longrightarrow R_1; s_{i1} \mapsto s_{i1}s''_{i11}, s'_{ij} \mapsto s'_{ij}s''_{ij1}.$$

Let $\pi_{01} : (\text{Spec } R_2, N_2) \longrightarrow (\text{Spec } R_1, N_1)$ over $(\overline{Y}, M_{\overline{Y}})$ defined by the natural inclusion $R_1 \xrightarrow{\subset} R_2$, and let π_{02} (resp. π_{12}) be the morphism $(\text{Spec } R_2, N_2) \longrightarrow (\text{Spec } R_1, N_1)$ over $(\overline{Y}, M_{\overline{Y}})$ defined by

$$\begin{aligned} R_1 &\longrightarrow R_2; s_{i1} \mapsto s_{i1}, s'_{ij} \mapsto s'_{ij}, s''_{ij1} \mapsto s''_{ij1} s''_{ij2}. \\ (\text{resp. } R_1 &\longrightarrow R_2; s_{i1} \mapsto s_{i1} s''_{i11}, s'_{ij} \mapsto s'_{ij} s''_{ij1}, s''_{ij1} \mapsto s''_{ij2}). \end{aligned}$$

Then we have $\pi_0 \circ \pi_{01} = \pi_0 \circ \pi_{02} (=:\varpi_0)$, $\pi_0 \circ \pi_{12} = \pi_1 \circ \pi_{01} (=:\varpi_1)$, $\pi_1 \circ \pi_{02} = \pi_1 \circ \pi_{12} (=:\varpi_2)$.

Let us prove that $(\text{Spec } R_m, N_m)$ is the $(m+1)$ -fold fiber product of $(\text{Spec } R_0, N_0)$ over $(\overline{Y}, M_{\overline{Y}})$ for $m = 1, 2$. Let us put $Q_1 := \mathbb{N}^r \oplus_{n, \mathbb{N}^r, n} \mathbb{N}^r$, $Q_2 := \mathbb{N}^r \oplus_{n, \mathbb{N}, n} \mathbb{N}^r \oplus_{n, \mathbb{N}^r, n} \mathbb{N}^r$ and denote by $Q_1^{\text{sat}}, Q_2^{\text{sat}}$ their saturation. Then we have the isomorphism $\mathbb{N}^r \oplus (\mathbb{Z}/n\mathbb{Z})^r \longrightarrow Q_1^{\text{sat}}$ as a special case of (2.26) and the isomorphism

$$\begin{aligned} Q_2^{\text{sat}} &= (\mathbb{N}^r \oplus_{n, \mathbb{N}^r, n} Q_1^{\text{sat}})^{\text{sat}} = (\mathbb{N}^r \oplus_{n, \mathbb{N}^r, n} \mathbb{N}^r \oplus (\mathbb{Z}/n\mathbb{Z})^r)^{\text{sat}} \\ &= Q_1^{\text{sat}} \oplus (\mathbb{Z}/n\mathbb{Z})^r = \mathbb{N}^r \oplus (\mathbb{Z}/n\mathbb{Z})^r \oplus (\mathbb{Z}/n\mathbb{Z})^r. \end{aligned}$$

Let $\alpha_{11} : \mathbb{Z}[Q_1] \longrightarrow R_0 \otimes_R R_0$, $\alpha_{12} : \mathbb{Z}[Q_2] \longrightarrow R_0 \otimes_R R_0 \otimes_R R_0$ be the homomorphism induced by α_1 . Then we can calculate the ring of global sections of the 2-fold fiber product $(\text{Spec } R_0, N_0)$ over $(\overline{Y}, M_{\overline{Y}})$ as follows:

$$\begin{aligned} &(R_0 \otimes_R R_0) \otimes_{\alpha_{11}, \mathbb{Z}[Q_1]} \mathbb{Z}[Q_1^{\text{sat}}] \\ &= (R[s''_{ij1}]_{1 \leq i \leq r, 1 \leq j \leq c} / ((s''_{ij1})^n - t_{ij}) \otimes_{\alpha_{11}, \mathbb{Z}[Q_1]} \mathbb{Z}[Q_1^{\text{sat}}]) \\ &= (R[s''_{ij1}]_{1 \leq i \leq r, 1 \leq j \leq c} / ((s''_{ij1})^n - 1) \quad (s''_{ij1} = s''_{ij1} s''_{ij1}^{-1}) \\ &= R_1. \end{aligned}$$

Moreover, it is easy to see that the log structure on the 2-fold fiber product of $(\text{Spec } R_0, N_0)$ over $(\overline{Y}, M_{\overline{Y}})$, being equal to the log structure associated to $Q_1^{\text{sat}} \longrightarrow R_1$ induced by the above diagram, is equal to the log structure associated to $\mathbb{N}^r \hookrightarrow Q_1^{\text{sat}} \longrightarrow R_1$, that is, the log structure N_1 . So $(\text{Spec } R_1, N_1)$ is the desired 2-fold fiber product. Similarly, we can calculate the ring of global sections of the 3-fold fiber product $(\text{Spec } R_0, N_0)$ over $(\overline{Y}, M_{\overline{Y}})$ as follows:

$$\begin{aligned} &(R_0 \otimes_R R_0 \otimes_R R_0) \otimes_{\alpha_{12}, \mathbb{Z}[Q_2]} \mathbb{Z}[Q_2^{\text{sat}}] \\ &= (R[s''_{ijl}]_{1 \leq i \leq r, 1 \leq j \leq c, l=1,2} / ((s''_{ijl})^n - t_{ij}) \otimes_{\alpha_{12}, \mathbb{Z}[Q_2]} \mathbb{Z}[Q_2^{\text{sat}}]) \\ &= (R[s''_{ijl}]_{1 \leq i \leq r, 1 \leq j \leq c, l=1,2} / ((s''_{ijl})^n - 1) \quad (s''_{ij1} = s''_{ij1} s''_{ij1}^{-1}, s''_{ij2} = s''_{ij2} s''_{ij1}^{-1}) \\ &= R_2. \end{aligned}$$

Moreover, it is easy to see that the log structure on the 3-fold fiber product of $(\text{Spec } R_0, N_0)$ over $(\overline{Y}, M_{\overline{Y}})$, being equal to the log structure associated to $Q_2^{\text{sat}} \longrightarrow R_2$ induced by the above diagram, is equal to the log structure associated to $\mathbb{N}^r \hookrightarrow Q_2^{\text{sat}} \longrightarrow R_2$, that is, the log structure N_2 . So $(\text{Spec } R_2, N_2)$ is the desired 3-fold fiber product. Hence, by using $\psi_0^{-1} \circ \pi_0$ and $\psi_0^{-1} \circ \pi_1$, we can define the isomorphism

$\psi_1 : (\text{Spec } R_1, N_1) \xrightarrow{=} (\overline{Y}_1, M_{\overline{Y}_1})$ over $(\overline{Y}, M_{\overline{Y}})$, and by using $\psi_0^{-1} \circ \varpi_0, \psi_0^{-1} \circ \varpi_1$ and $\psi_0^{-1} \circ \varpi_2$, we can define the isomorphism $\psi_2 : (\text{Spec } R_1, N_1) \xrightarrow{=} (\overline{Y}_1, M_{\overline{Y}_1})$ over $(\overline{Y}, M_{\overline{Y}})$. We can also check from the above concrete descriptions that, by the identification via ψ_m 's, the morphisms π_i ($i = 0, 1$), π_{ij} ($0 \leq i < j \leq 2$) correspond to the projections between $(\overline{Y}_m, M_{\overline{Y}_m})$'s. Recall that $(\text{Spec } R_0, N_0)$ admits the action of $\mu_n^r \times \mu_n^{r(c-1)} = \mu_n^{rc}$ (the action on s_{i1} 's and s'_{ij} 's). By the above definition of R_1 and R_2 , we see the isomorphisms $\text{Spec } R_m \cong \text{Spec } R_0 \times \mu_n^{rcm}$, and via this isomorphisms, the morphisms π_i ($i = 0, 1$), π_{ij} ($0 \leq i < j \leq 2$) are described by

$$\begin{aligned} \pi_0 : (y, \eta) &\mapsto y, \quad \pi_1 : (y, \eta) \mapsto y^\eta, \\ \pi_{01} : (y, \eta, \zeta) &\mapsto (y, \eta), \quad \pi_{02} : (y, \eta, \zeta) \mapsto (y, \eta\zeta), \quad \pi_{12} : (y, \eta, \zeta) \mapsto (y^\eta, \zeta), \end{aligned}$$

where $y \in \text{Spec } R_0, \eta, \zeta \in \mu_n^{rc}$ and the action of η is denoted by $y \mapsto y^\eta$. By this description, we see that the diagram $\text{Spec } R_\bullet \rightarrow [\text{Spec } R_0 / \mu_n^{rc}]$ is the 2-truncated etale Čech hypercovering associated to the quotient map $\text{Spec } R_0 \rightarrow [\text{Spec } R_0 / \mu_n^{rc}]$. Using the identification by ψ_m 's, we conclude that the diagram $\overline{Y}_\bullet \rightarrow [\overline{Y}_0 / \mu_n^{rc}]$ is the 2-truncated etale Čech hypercovering associated to the quotient map $\overline{Y}_0 \rightarrow [\overline{Y}_0 / \mu_n^{rc}]$, that is, we have shown the assertion (1).

Finally we prove that the morphism (2.25) satisfies the assertion in (2). For $m = 0$, this follows from the definition of the morphism (2.25) (see the diagram (2.27)). For $m = 1, 2$, this follows from the fact that the projections $(\overline{Y}_m, M_{\overline{Y}_m}) \rightarrow (\overline{Y}_0, M_{\overline{Y}_0})$ are strict (which follows from the same assertion for π_i 's and π_{ij} 's which can be seen easily) and the assertion in the case $m = 0$. So we are done. \square

In view of Proposition 2.20, we make the following definition.

Definition 2.21. *Let (\overline{X}, Z) , n and $(\overline{Y}, J, \{t_{ij}\})$ be as above. Then we call the 2-truncated simplicial log scheme $(\overline{Y}_\bullet, M_{\overline{Y}_\bullet})$ constructed above a simplicial resolution of $\overline{Y} \times_{\overline{X}} (\overline{X}, Z)^{1/n}$ associated to the multichart $(\overline{Y}, J, \{t_{ij}\})$.*

Using this, we define the notion of a bisimplicial resolution as follows:

Definition 2.22. *Let (\overline{X}, Z) and n be as above and let $(\overline{X}_0, \{t_i\}_{i=1}^r)$ be a chart for (\overline{X}, Z) . Then:*

- (1) *We define the 2-truncated simplicial multichart $(\overline{X}_\bullet, J_\bullet, \{t_{ij}^{(\bullet)}\})$ associated to $(\overline{X}_0, \{t_i\}_{i=1}^r)$ as follows: For $m = 0, 1, 2$, let \overline{X}_m be the $(m+1)$ -fold fiber product of \overline{X}_0 over \overline{X} , and let us define $J_m := \{0, \dots, m\}$. If we denote the projections $\overline{X}_m \rightarrow \overline{X}_0$ by π_a ($0 \leq a \leq m$), $t_{ij}^{(a)}$ is defined to be $\pi_a^*(t_i)$. We call the 2-truncated simplicial log scheme $(\overline{X}_\bullet, M_{\overline{X}_\bullet}) := (\overline{X}_\bullet, M_{\overline{X}|\overline{X}_\bullet})$ the simplicial semi-resolution of $(\overline{X}, Z)^{1/n}$ associated to the chart $(\overline{X}_0, \{t_i\}_{i=1}^r)$.*
- (2) *Keep the notation of (1). For $m = 0, 1, 2$, we have the simplicial resolution $(\overline{X}_{m\bullet}, M_{\overline{X}_{m\bullet}})$ of $\overline{X}_m \times_{\overline{X}} (\overline{X}, Z)^{1/n}$ associated to the multichart $(\overline{X}_m, J_m, \{t_{ij}^{(m)}\})$,*

and by functoriality, they form a $(2, 2)$ -truncated bisimplicial log scheme $(\overline{X}_{\bullet\bullet}, M_{\overline{X}_{\bullet\bullet}})$. We call it the bisimplicial resolution of $(\overline{X}, Z)^{1/n}$ associated to the chart $(\overline{X}_0, \{t_i\}_{i=1}^r)$.

Let the notation be as in Definition 2.22. Then, by Proposition 2.20, we have 2-truncated etale Čech hypercoverings $\overline{X}_{m\bullet} \rightarrow \overline{X}_m \times_{\overline{X}} (\overline{X}, Z)^{1/n}$ for $m = 0, 1, 2$ and so we have the diagram

$$(2.29) \quad \overline{X}_{\bullet\bullet} \rightarrow \overline{X}_{\bullet} \times_{\overline{X}} (\overline{X}, Z)^{1/n} \rightarrow (\overline{X}, Z)^{1/n},$$

where the first and the second morphisms are both 2-truncated etale Čech hypercoverings. So we have equivalences

$$(2.30) \quad \mathrm{Isoc}((\overline{X}, Z)^{1/n}) \xrightarrow{=} \mathrm{Isoc}(\overline{X}_{\bullet} \times_{\overline{X}} (\overline{X}, Z)^{1/n}) \xrightarrow{=} \mathrm{Isoc}(\overline{X}_{\bullet\bullet}).$$

We also have equivalences

$$(2.31) \quad F\text{-}\mathrm{Isoc}((\overline{X}, Z)^{1/n}) \xrightarrow{=} F\text{-}\mathrm{Isoc}(\overline{X}_{\bullet} \times_{\overline{X}} (\overline{X}, Z)^{1/n}) \xrightarrow{=} F\text{-}\mathrm{Isoc}(\overline{X}_{\bullet\bullet}),$$

$$(2.32) \quad F\text{-}\mathrm{Isoc}((\overline{X}, Z)^{1/n})^{\circ} \xrightarrow{=} F\text{-}\mathrm{Isoc}(\overline{X}_{\bullet} \times_{\overline{X}} (\overline{X}, Z)^{1/n})^{\circ} \xrightarrow{=} F\text{-}\mathrm{Isoc}(\overline{X}_{\bullet\bullet})^{\circ}.$$

We also have the log version: Note that we have the fine log structure $M_{(\overline{X}, Z)^{1/n}}$ on $(\overline{X}, Z)^{1/n}$ defined by the projection $(\overline{X}, Z)^{1/n} \rightarrow [\mathbb{A}_k^r / \mathbb{G}_{m,k}^r]$ via [34, 5.13]. Then, by Proposition 2.20(2), we can endow the diagram (2.29) with log structures and form the diagram

$$(2.33) \quad (\overline{X}_{\bullet\bullet}, M_{\overline{X}_{\bullet\bullet}}) \rightarrow \overline{X}_{\bullet} \times_{\overline{X}} ((\overline{X}, Z)^{1/n}, M_{(\overline{X}, Z)^{1/n}}) \rightarrow ((\overline{X}, Z)^{1/n}, M_{(\overline{X}, Z)^{1/n}}),$$

where the first and the second morphisms are both 2-truncated strict etale Čech hypercoverings. So we have equivalences

$$(2.34) \quad \mathrm{Isoc}^{\log}((\overline{X}, Z)^{1/n}, M_{(\overline{X}, Z)^{1/n}}) \xrightarrow{=} \mathrm{Isoc}^{\log}(\overline{X}_{\bullet} \times_{\overline{X}} (\overline{X}, Z)^{1/n})$$

$$\xrightarrow{=} \mathrm{Isoc}^{\log}(\overline{X}_{\bullet\bullet}, M_{\overline{X}_{\bullet\bullet}}),$$

$$(2.35) \quad F\text{-}\mathrm{Isoc}^{\log}((\overline{X}, Z)^{1/n}, M_{(\overline{X}, Z)^{1/n}}) \xrightarrow{=} F\text{-}\mathrm{Isoc}^{\log}(\overline{X}_{\bullet} \times_{\overline{X}} (\overline{X}, Z)^{1/n})$$

$$\xrightarrow{=} F\text{-}\mathrm{Isoc}^{\log}(\overline{X}_{\bullet\bullet}, M_{\overline{X}_{\bullet\bullet}}).$$

We also have the log version with condition on exponents: Note that, for any $1 \leq i \leq r$, the morphism

$$(\overline{X}, Z)^{1/n} \rightarrow [\mathbb{A}_k^r / \mathbb{G}_{m,k}^r] \xrightarrow{i\text{-th proj.}} [\mathbb{A}_k / \mathbb{G}_{m,k}]$$

induces a fine log structure on $(\overline{X}, Z)^{1/n}$ which we denote by $M_{(\overline{X}, Z)^{1/n}, i}$. In the notation of Proposition 2.20 and its proof, the pull-back of $M_{(\overline{X}, Z)^{1/n}}$ and $M_{(\overline{X}, Z)^{1/n}, i}$ by the morphism

$$\overline{Y}_0 \rightarrow [\overline{Y} / \mu_n^{rc}] \xrightarrow{(2.25)} \overline{Y} \times_{\overline{X}} (\overline{X}, Z)^{1/n} \rightarrow (\overline{X}, Z)^{1/n}$$

is associated to the simple normal crossing divisor $\bigcup_{i=1}^r Z_{0,i}$ and its subdivisor $Z_{0,i}$, respectively (see the diagram (2.27)). So the log structure $M_{(\overline{X}, Z)^{1/n}}$ satisfies the condition $(*)$ in Section 2.1 and $\{M_{(\overline{X}, Z)^{1/n}, i}\}_{i=1}^r$ is a decomposition of $M_{(\overline{X}, Z)^{1/n}}$, which also induces the decomposition $\{M_{\overline{X} \bullet \times_{\overline{X}} (\overline{X}, Z)^{1/n}, i}\}_i$ of $M_{\overline{X} \bullet \times_{\overline{X}} (\overline{X}, Z)^{1/n}} := M_{(\overline{X}, Z)^{1/n}}|_{\overline{X} \bullet \times_{\overline{X}} (\overline{X}, Z)^{1/n}}$ and the decomposition $\{M_{\overline{X} \bullet \bullet, i}\}_i$ of $M_{\overline{X} \bullet \bullet}$. Hence we have the equivalence

$$(2.36) \quad \begin{aligned} & \text{Isoc}^{\log}((\overline{X}, Z)^{1/n}, M_{(\overline{X}, Z)^{1/n}})_{\Sigma(-ss)} \\ & \xrightarrow{=} \text{Isoc}^{\log}(\overline{X} \bullet \times_{\overline{X}} (\overline{X}, Z)^{1/n}, M_{\overline{X} \bullet \times_{\overline{X}} (\overline{X}, Z)^{1/n}})_{\Sigma(-ss)} \\ & \xrightarrow{=} \text{Isoc}^{\log}(\overline{X} \bullet \bullet, M_{\overline{X} \bullet \bullet})_{\Sigma(-ss)} \end{aligned}$$

for $\Sigma = \prod_{i=1}^r \Sigma_i \subseteq \mathbb{Z}_p^r$, by Corollary 2.13.

Finally, we explain the functoriality of the categories of isocrystals on the stack of roots $(\overline{X}, Z)^{1/n}$ with respect to n . By definition, we have the morphism

$$(2.37) \quad \begin{aligned} (\overline{X}, Z)^{1/n'} &= \overline{X} \times_{[\mathbb{A}_k^r/\mathbb{G}_{m,k}^r], n'} [\mathbb{A}_k^r/\mathbb{G}_{m,k}^r] \\ &\xrightarrow{\text{id} \times (n'/n)} \overline{X} \times_{[\mathbb{A}_k^r/\mathbb{G}_{m,k}^r], n} [\mathbb{A}_k^r/\mathbb{G}_{m,k}^r] = (\overline{X}, Z)^{1/n} \end{aligned}$$

for n, n' with $n \mid n'$, where (n'/n) denotes the (n'/n) -th power map. Using the morphisms (2.37) for all n, n' with $n \mid n'$, we can form the limit

$$\varinjlim_{(n,p)=1} \text{Isoc}((\overline{X}, Z)^{1/n}), \quad \varinjlim_{(n,p)=1} F\text{-Isoc}((\overline{X}, Z)^{1/n}), \quad \varinjlim_{(n,p)=1} F\text{-Isoc}((\overline{X}, Z)^{1/n})^\circ.$$

We also have the log version: For a positive integer a , the a -th power morphism $[\mathbb{A}_k^r/\mathbb{G}_{m,k}^r] \rightarrow [\mathbb{A}_k^r/\mathbb{G}_{m,k}^r]$ naturally induces the morphism of fine log algebraic stacks $([\mathbb{A}_k^r/\mathbb{G}_{m,k}^r], M_{[\mathbb{A}_k^r/\mathbb{G}_{m,k}^r]}) \rightarrow ([\mathbb{A}_k^r/\mathbb{G}_{m,k}^r], M_{[\mathbb{A}_k^r/\mathbb{G}_{m,k}^r]})$ by [34, pp.780 – 781] and so the morphism (2.37) is enriched to a morphism of fine log algebraic stacks

$$(2.38) \quad ((\overline{X}, Z)^{1/n'}, M_{(\overline{X}, Z)^{1/n'}}) \rightarrow ((\overline{X}, Z)^{1/n}, M_{(\overline{X}, Z)^{1/n}}).$$

So we have also the limit

$$\varinjlim_{(n,p)=1} \text{Isoc}^{\log}((\overline{X}, Z)^{1/n}, M_{(\overline{X}, Z)^{1/n}}), \quad \varinjlim_{(n,p)=1} F\text{-Isoc}^{\log}((\overline{X}, Z)^{1/n}, M_{(\overline{X}, Z)^{1/n}}).$$

We also have the log version with exponent condition: Let us take $\Sigma := \prod_{i=1}^r \Sigma_i \subseteq \mathbb{Z}_p^r$. Fix for the moment two positive integers n, n' with $n \mid n'$. Let us fix a multichart $(\overline{Y}, J, \{t_{ij}\})$ with $J = \{1\}$ and construct $(\overline{Y}_0, M_{\overline{Y}_0}) \cong (\text{Spec } R_0, N_0)$, $s_{i1} \in R_0$, $Z_{0,i} = \{s_{i1} = 0\}$ ($1 \leq i \leq r$) as in the proof of Proposition 2.20 for n and n' : We denote $(\overline{Y}_0, M_{\overline{Y}_0}) \cong (\text{Spec } R_0, N_0)$, $s_{i1}, Z_{0,i}$ for n (resp. n') by $(\overline{Y}_0^{(n)}, M_{\overline{Y}_0^{(n)}}) \cong (\text{Spec } R_0^{(n)}, N_0^{(n)})$, $s_{i1}^{(n)}, Z_{0,i}^{(n)}$ (resp. $(\overline{Y}_0^{(n')}, M_{\overline{Y}_0^{(n')}}) \cong (\text{Spec } R_0^{(n')}, N_0^{(n')})$, $s_{i1}^{(n')}, Z_{0,i}^{(n')}$). Then we can define the log etale morphism

$$(2.39) \quad (\overline{Y}_0^{(n')}, M_{\overline{Y}_0^{(n')}}) \rightarrow (\overline{Y}_0^{(n)}, M_{\overline{Y}_0^{(n)}})$$

fitting into the diagram

$$\begin{array}{ccc}
(\overline{Y}_0^{(n')}, M_{\overline{Y}_0^{(n')}}) & \xrightarrow{(2.39)} & (\overline{Y}_0^{(n)}, M_{\overline{Y}_0^{(n)}}) \\
\downarrow & & \downarrow \\
((\overline{X}, Z)^{1/n'}, M_{(\overline{X}, Z)^{1/n'}}) & \xrightarrow{(2.38)} & ((\overline{X}, Z)^{1/n}, M_{(\overline{X}, Z)^{1/n}})
\end{array}$$

by $s_{i1}^{(n)} \mapsto s_{i1}^{(n')^{n'/n}}$. Then, by Proposition 1.22, the upper horizontal arrow induces the functor

$$\mathrm{Isoc}^{\log}(\overline{Y}^{(n)}, M_{\overline{Y}^{(n)}})_{n\Sigma(-\mathrm{ss})} \longrightarrow \mathrm{Isoc}^{\log}(\overline{Y}^{(n')}, M_{\overline{Y}^{(n')}})_{n'\Sigma(-\mathrm{ss})},$$

where the categories are defined with respect to the decomposition $\{Z_{0,i}^{(n)}\}_i$ of $\bigcup_i Z_{0,i}^{(n)}$ and the decomposition $\{Z_{0,i}^{(n')}\}_i$ of $\bigcup_i Z_{0,i}^{(n')}$. From this, we see the existence of the canonical functor

$$\mathrm{Isoc}^{\log}((\overline{X}, Z)^{1/n}, M_{(\overline{X}, Z)^{1/n}})_{n\Sigma(-\mathrm{ss})} \longrightarrow \mathrm{Isoc}^{\log}((\overline{X}, Z)^{1/n'}, M_{(\overline{X}, Z)^{1/n'}})_{n'\Sigma(-\mathrm{ss})},$$

and this functor for all n, n' with $n \mid n'$ induces the limit

$$\varinjlim_{(n,p)=1} \mathrm{Isoc}^{\log}((\overline{X}, Z)^{1/n}, M_{(\overline{X}, Z)^{1/n}})_{n\Sigma(-\mathrm{ss})}.$$

2.4 Second stacky equivalence

In this subsection, we prove the equivalence (0.9) and related equivalences. In this subsection, let $X \hookrightarrow \overline{X}$ be an open immersion of connected smooth varieties over k with $\overline{X} \setminus X =: Z = \bigcup_{i=1}^r Z_i$ a simple normal crossing divisor (each Z_i being irreducible). Let v_i ($1 \leq i \leq r$) be the discrete valuation $k(X)$ corresponding to the generic point of Z_i .

First we define a functor relating the right hand sides of (0.8) and (0.9) in a slightly generalized form.

Proposition 2.23. *Let $(X, \overline{X}), Z$ be as above. Let \mathcal{G}_X^t be the category of finite etale Galois tame covering (tamely ramified at v_i ($1 \leq i \leq r$)) and for $Y \rightarrow X$ in \mathcal{G}_X^t , let $G_Y := \mathrm{Aut}(Y/X)$ and let $\overline{Y}^{\mathrm{sm}}$ be the smooth locus of the normalization \overline{Y} of \overline{X} in $k(Y)$. Then we have the canonical functors*

$$(2.40) \quad \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} F\text{-Isoc}([\overline{Y}^{\mathrm{sm}}/G_Y]) \longrightarrow \varinjlim_{(n,p)=1} F\text{-Isoc}((\overline{X}, Z)^{1/n})$$

$$(2.41) \quad \varinjlim_{(n,p)=1} \mathrm{Isoc}((\overline{X}, Z)^{1/n}) \longrightarrow \mathrm{Isoc}^\dagger(X, \overline{X})$$

which make the following diagram commutative:

$$(2.42) \quad \begin{array}{ccc} \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} F\text{-Isoc}([\overline{Y}^{\text{sm}}/G_Y]) & \xrightarrow{(2.40)} & \varinjlim_{(n,p)=1} F\text{-Isoc}((\overline{X}, Z)^{1/n}) \\ (2.17) \downarrow & & \downarrow F\text{-(2.41)} \\ F\text{-Isoc}^\dagger(X, \overline{X}) & \xlongequal{\quad} & F\text{-Isoc}^\dagger(X, \overline{X}). \end{array}$$

When X is a curve, we have also the functor

$$(2.43) \quad \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} \text{Isoc}([\overline{Y}^{\text{sm}}/G_Y]) \longrightarrow \varinjlim_{(n,p)=1} \text{Isoc}((\overline{X}, Z)^{1/n})$$

with $F\text{-(2.43)} = (2.40)$ which makes the following diagram commutative:

$$(2.44) \quad \begin{array}{ccc} \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} \text{Isoc}([\overline{Y}^{\text{sm}}/G_Y]) & \xrightarrow{(2.43)} & \varinjlim_{(n,p)=1} \text{Isoc}((\overline{X}, Z)^{1/n}) \\ (2.19) \downarrow & & \downarrow (2.41) \\ \text{Isoc}^\dagger(X, \overline{X}) & \xlongequal{\quad} & \text{Isoc}^\dagger(X, \overline{X}). \end{array}$$

Proof. First let us take an object $Y \rightarrow X$ in \mathcal{G}_X^t and construct a functor

$$(2.45) \quad F\text{-Isoc}([\overline{Y}^{\text{sm}}/G_Y]) \longrightarrow F\text{-Isoc}((\overline{X}, Z)^{1/n})$$

for some n . Let $M_{\overline{X}}, M_{\overline{Y}}$ be the log structure on $\overline{X}, \overline{Y}$ defined by $Z, \overline{Y} \setminus Y$, respectively. Then, by [13, 4.7(c), 7.6], the morphism $Y \rightarrow X$ naturally induces a finite Kummer log etale morphism $f : (\overline{Y}, M_{\overline{Y}}) \rightarrow (\overline{X}, M_{\overline{X}})$ of fs log schemes.

In this proof, we follow the convention that fiber products of fs log schemes are always taken in the category of fs log schemes. For $m = 0, 1, 2$, let $(\overline{Y}_m, M_{\overline{Y}_m})$ be the $(m+1)$ -fold fiber product of $(\overline{Y}, M_{\overline{Y}})$ over $(\overline{X}, M_{\overline{X}})$. Then we have $(\overline{Y}_m, M_{\overline{Y}_m}) \cong (\overline{Y}, M_{\overline{Y}}) \times_{\overline{Y}}^m \overline{Y}$ (one can see it by using [13, 7.6] and noting the isomorphism $Y \times_X Y \cong Y \times G$) and so we have the equivalence

$$(2.46) \quad \text{Isoc}([\overline{Y}^{\text{sm}}/G_Y]) \cong \text{Isoc}(\overline{Y}_\bullet^{\text{sm}}),$$

where $\overline{Y}_m^{\text{sm}}$ is the smooth locus of \overline{Y}_m and $\overline{Y}_\bullet^{\text{sm}}$ is the 2-truncated simplicial scheme formed by $\overline{Y}_m^{\text{sm}}$ ($m = 0, 1, 2$). (Note that $\overline{Y}_\bullet^{\text{sm}}$ here is isomorphic to $\overline{Y}_\bullet^{\text{sm}}$ in the proof of Theorem 2.17.)

By [43, 2.2, 2.6], f is of n -Kummer type for some $n \in \mathbb{N}$ which is prime to p in the terminology of [43]. Fix one such n . Take a chart $(\overline{X}_0, \{t_i\}_{1 \leq i \leq r})$ for (\overline{X}, Z) in the sense of Section 2.3, and let $(\overline{X}_\bullet, M_{\overline{X}_\bullet}), (\overline{X}_{\bullet\bullet}, M_{\overline{X}_{\bullet\bullet}})$ be the simplicial semi-resolution, the bisimplicial resolution of $(\overline{X}, Z)^{1/n}$ associated to $(\overline{X}_0, \{t_i\}_{1 \leq i \leq r})$ respectively (see Definition 2.22). Then we have the equivalence of categories (2.31). Let \overline{X}' be the image of \overline{Y}^{sm} in \overline{X} and let us put $(\overline{X}'_{lm}, M_{\overline{X}'_{lm}}) := \overline{X}' \times_{\overline{X}} (\overline{X}_{lm}, M_{\overline{X}_{lm}})$. For $k, l, m \in \{0, 1, 2\}$, let us put $(\overline{U}_{klm}, M_{\overline{U}_{klm}}) := (\overline{Y}_k^{\text{sm}}, M_{\overline{Y}_k} |_{\overline{Y}_k^{\text{sm}}}) \times_{(\overline{X}', M_{\overline{X}'})} (\overline{X}'_{lm}, M_{\overline{X}'_{lm}})$.

$M_{\overline{X}'_{lm}}$) and let $g_{klm} : (\overline{U}_{klm}, M_{\overline{U}_{klm}}) \longrightarrow (\overline{X}'_{lm}, M_{\overline{X}'_{lm}})$, $h_{klm} : (\overline{U}_{klm}, M_{\overline{U}_{klm}}) \longrightarrow (\overline{Y}_k^{\text{sm}}, M_{\overline{Y}_k} |_{\overline{Y}_k^{\text{sm}}})$ be the projections. Then, since f is of n -Kummer type, g_{0lm} is a strict finite etale Galois morphism with Galois group G_Y by [43, 2.5] and so $g_{\bullet lm}$ is a 2-truncated strict etale Čech hypercovering. Let $\overline{U}_{\bullet\bullet\bullet}$ be the $(2, 2, 2)$ -truncated trisimplicial scheme formed by \overline{U}_{klm} 's. Then, by etale descent and [47, 3.1], we have the equivalences

$$(2.47) \quad \text{Isoc}(\overline{X}'_{\bullet\bullet}) \xrightarrow{=} \text{Isoc}(\overline{U}_{\bullet\bullet\bullet}),$$

$$(2.48) \quad F\text{-Isoc}(\overline{X}_{\bullet\bullet}) \xrightarrow{=} F\text{-Isoc}(\overline{X}'_{\bullet\bullet}).$$

Now we define the functor (2.45) as the composite

$$(2.49) \quad \begin{aligned} F\text{-Isoc}([\overline{Y}^{\text{sm}}/G_Y]) &\xrightarrow{F\text{-(2.46)}} F\text{-Isoc}(\overline{Y}_{\bullet}^{\text{sm}}) \\ &\xrightarrow{h_{\bullet\bullet\bullet}^*} F\text{-Isoc}(\overline{U}_{\bullet\bullet\bullet}) \\ &\xrightarrow{F\text{-(2.47)}^{-1}} F\text{-Isoc}(\overline{X}'_{\bullet\bullet}) \\ &\xrightarrow{(2.48)^{-1}} F\text{-Isoc}(\overline{X}_{\bullet\bullet}) \\ &\xrightarrow{(2.31)^{-1}} F\text{-Isoc}((\overline{X}, Z)^{1/n}). \end{aligned}$$

Then we see that this functor induces the desired functor (2.40). When X is a curve, we can define the functor (2.43) with $F\text{-(2.43)} = (2.40)$ in the same way as above: The only problem without Frobenius structure is that we do not know the analogue of the equivalence (2.48) in general, but this does not cause any problem when \overline{X} is a curve because we have $\overline{X} = \overline{X}'$ in this case.

Next we would like to define the functor (2.41) as the one induced by the composite

$$(2.50) \quad \begin{aligned} \text{Isoc}((\overline{X}, Z)^{1/n}) &\xrightarrow{(2.30)} \text{Isoc}(\overline{X}_{\bullet\bullet}) \\ &\longrightarrow \text{Isoc}^\dagger(X_{\bullet\bullet}, \overline{X}_{\bullet\bullet}) \quad (X_{\bullet\bullet} := X \times_{\overline{X}} \overline{X}_{\bullet\bullet}) \\ &\xleftarrow{=} \text{Isoc}^\dagger(X_{\bullet}, \overline{X}_{\bullet}) \quad (X_{\bullet} := X \times_{\overline{X}} \overline{X}_{\bullet}) \\ &\xleftarrow{=} \text{Isoc}^\dagger(X, \overline{X}). \end{aligned}$$

In order that the functor is well-defined, we should prove that the third arrow in the above composite is an equivalence. (It is rather easy to see that the fourth arrow is an equivalence because $\overline{X}_{\bullet} \longrightarrow \overline{X}$ is a 2-truncated etale Čech hypercovering. See [46, 5.1] for example.) This can be shown in the following way. (The proof here is analogous to the proof of (2.13) in the proof of Theorem 2.17.) We may enlarge k in order that k contains a primitive n -th root of unity and it suffices to prove the equivalence of the restriction functor

$$(2.51) \quad \text{Isoc}^\dagger(X_l, \overline{X}_l) \longrightarrow \text{Isoc}^\dagger(X_{l\bullet}, \overline{X}_{l\bullet}).$$

In this case, we have $\overline{X}_{lm} \cong \overline{X}_{l0} \times \mu_n(k)^{r(l+1)m}$ by Proposition (2.20)(1). So the right hand side is the category of objects in $\text{Isoc}^\dagger(X_{l0}, \overline{X}_{l0})$ endowed with equivariant $\mu_n(k)^{r(l+1)}$ -action. Then, if we denote the projection $\overline{X}_{l0} \rightarrow \overline{X}_l$ by π , we have the functor

$$(2.52) \quad \text{Isoc}^\dagger(X_{l\bullet}, \overline{X}_{l\bullet}) \longrightarrow \text{Isoc}^\dagger(X_l, \overline{X}_l); \quad \mathcal{E} \mapsto (\pi_* \mathcal{E})^{\mu_n(k)^{r(l+1)}},$$

where π_* is the push-out functor defined by Tsuzuki [49]. By [49] and [21, 2.6.8], we have the trace morphisms $(\pi_* \pi^* \mathcal{E})^{\mu_n(k)^{r(l+1)}} \rightarrow \mathcal{E}$, $\pi^*((\pi_* \mathcal{E})^{\mu_n(k)^{r(l+1)}}) \rightarrow \mathcal{E}$, and they are isomorphic in $\text{Isoc}(X_l)$, $\text{Isoc}(X_{l\bullet})$ by etale descent. Since $\text{Isoc}^\dagger(X_l, \overline{X}_l) \rightarrow \text{Isoc}(X_l)$, $\text{Isoc}^\dagger(X_{l\bullet}, \overline{X}_{l\bullet}) \rightarrow \text{Isoc}(X_{l\bullet})$ are exact and faithful, they are actually isomorphic. Hence (2.52) is a quasi-inverse of (2.51) and so (2.51) is an equivalence. So the functor (2.41) is defined.

We prove the commutativity of the diagram (2.42). By definition, (2.17) is defined as the composite

$$(2.53) \quad \begin{aligned} F\text{-Isoc}([\overline{Y}^{\text{sm}}/G_Y]) &\longrightarrow F\text{-Isoc}(\overline{Y}_\bullet^{\text{sm}}) \\ &\longrightarrow F\text{-Isoc}^\dagger(Y_\bullet, \overline{Y}_\bullet^{\text{sm}}) \\ &\stackrel{(2.13)}{\longleftarrow} F\text{-Isoc}^\dagger(X, \overline{X}') \stackrel{[47, 3.1]}{\longleftarrow} F\text{-Isoc}^\dagger(X, \overline{X}). \end{aligned}$$

Noting the commutative square of equivalences

$$\begin{array}{ccc} F\text{-Isoc}^\dagger(X, \overline{X}) & \longrightarrow & F\text{-Isoc}^\dagger(X_{\bullet\bullet}, \overline{X}_{\bullet\bullet}) \\ \downarrow & & \downarrow \\ F\text{-Isoc}^\dagger(X, \overline{X}') & \longrightarrow & F\text{-Isoc}^\dagger(X_{\bullet\bullet}, \overline{X}'_{\bullet\bullet}) \end{array}$$

and the functoriality of restriction functors, we see that the third line in the diagram (2.53) is rewritten as follows:

$$\begin{aligned} F\text{-Isoc}^\dagger(Y_\bullet, \overline{Y}_\bullet^{\text{sm}}) &\xrightarrow{h_{\bullet\bullet}^*} F\text{-Isoc}^\dagger(U_{\bullet\bullet}, \overline{U}_{\bullet\bullet}) \\ &\stackrel{=}{\longleftarrow} F\text{-Isoc}^\dagger(X_{\bullet\bullet}, \overline{X}'_{\bullet\bullet}) \\ &\stackrel{=}{\longleftarrow} F\text{-Isoc}^\dagger(X_{\bullet\bullet}, \overline{X}_{\bullet\bullet}) \stackrel{=}{\longleftarrow} F\text{-Isoc}^\dagger(X, \overline{X}). \end{aligned}$$

(Here $U_{\bullet\bullet} := X \times_{\overline{X}} \overline{U}_{\bullet\bullet}$. The equivalence of the second arrow follows from the etale descent for the category of overconvergent isocrystals [46, 5.1].) Using this description and the functoriality of restriction functors, we see that (2.17) is rewritten as

$$\begin{aligned} F\text{-Isoc}([\overline{Y}^{\text{sm}}/G_Y]) &\longrightarrow F\text{-Isoc}(\overline{Y}_\bullet^{\text{sm}}) \\ &\xrightarrow{h_{\bullet\bullet}^*} F\text{-Isoc}(\overline{U}_{\bullet\bullet}) \\ &\longrightarrow F\text{-Isoc}(\overline{X}'_{\bullet\bullet}) \\ &\stackrel{=}{\longleftarrow} F\text{-Isoc}(\overline{X}_{\bullet\bullet}) \\ &\longrightarrow F\text{-Isoc}^\dagger(X_{\bullet\bullet}, \overline{X}_{\bullet\bullet}) \stackrel{=}{\longleftarrow} F\text{-Isoc}^\dagger(X, \overline{X}), \end{aligned}$$

and we see from definition that this is equal to the composite $F\text{-(2.41)} \circ (2.40)$. So we have proved the commutativity of the diagram (2.42). When X is a curve, we can prove the commutativity of the diagram (2.44) in the same way. (We do not have to use [47, 3.1] in this case because we have $\overline{X} = \overline{X}'$, $\overline{X}_{\bullet\bullet} = \overline{X}'_{\bullet\bullet}$ when X is a curve.) So we are done. \square

Remark 2.24. In the notation of Proposition 2.23, we have the log version (with exponent condition) of the functor (2.50) (for any $\Sigma = \prod_{i=1}^r \Sigma_i \subseteq \mathbb{Z}_p^r$):

$$\begin{aligned}
(2.54) \quad \text{Isoc}^{\log}((\overline{X}, Z)^{1/n}, M_{(\overline{X}, Z)^{1/n}})_{(\Sigma(-\text{ss}))} &\xrightarrow{(2.34), (2.36)} \text{Isoc}^{\log}(\overline{X}_{\bullet\bullet}, M_{\overline{X}_{\bullet\bullet}})_{(\Sigma(-\text{ss}))} \\
&\longrightarrow \text{Isoc}^{\dagger}(X_{\bullet\bullet}, \overline{X}_{\bullet\bullet}) \\
&\xleftarrow{=} \text{Isoc}^{\dagger}(X_{\bullet}, \overline{X}_{\bullet}) \\
&\xleftarrow{=} \text{Isoc}^{\dagger}(X, \overline{X}).
\end{aligned}$$

So we have the functor

$$(2.55) \quad \varinjlim_{(n,p)=1} \text{Isoc}^{\log}((\overline{X}, Z)^{1/n})_{(n\Sigma(-\text{ss}))} \longrightarrow \text{Isoc}^{\dagger}(X, \overline{X}),$$

which is the log version (with exponent condition) of (2.41). When Σ is (NID) and (NLD), the functor (2.54) (the version with the subscript Σ or $\Sigma\text{-ss}$) is fully faithful since the first arrow (2.36) is an equivalence and the second arrow is fully faithful by Theorem 1.17. (see also Definition 1.3.) Therefore, when Σ is (NRD) and (SNLD), the functor (2.55) (the version with the subscript $n\Sigma$ or $n\Sigma\text{-ss}$) is fully faithful by Lemma 1.4.

Remark 2.25. In this remark, we give a construction of the log version (with exponent condition) of (2.43) when X is a curve.

In the following, we follow the notation in the proof of Proposition 2.23. Then, in the same way as (2.49), we can define the functor

$$\begin{aligned}
(2.56) \quad \text{Isoc}^{\log}([\overline{Y}/G_Y], M_{[\overline{Y}/G_Y]}) &\longrightarrow \text{Isoc}^{\log}(\overline{Y}_{\bullet}, M_{\overline{Y}_{\bullet}}) \\
&\xrightarrow{h_{\bullet\bullet}^*} \text{Isoc}^{\log}(\overline{U}_{\bullet\bullet\bullet}, M_{\overline{U}_{\bullet\bullet\bullet}}) \\
&\xrightarrow{(2.47)^{-1}} \text{Isoc}^{\log}(\overline{X}_{\bullet\bullet}, M_{\overline{X}_{\bullet\bullet}}) \\
&\xrightarrow{(2.34)^{-1}} \text{Isoc}((\overline{X}, Z)^{1/n}, M_{(\overline{X}, Z)^{1/n}}).
\end{aligned}$$

(We have $\overline{Y}^{\text{sm}} = \overline{Y}$, $\overline{X}_{\bullet\bullet} = \overline{X}'_{\bullet\bullet}$ because X is a curve.) Also, let us note that, for a morphism $f : Y' \longrightarrow Y$ in \mathcal{G}_X^t , we have a morphism of log schemes $(\overline{Y}', M_{\overline{Y}'}) \longrightarrow (\overline{Y}, M_{\overline{Y}})$ which is equivariant with $G_{Y'} \longrightarrow G_Y$ ([13]). So the morphism $[\overline{Y}']/G_{Y'} \longrightarrow [\overline{Y}]/G_Y$ induced by f is enriched to a morphism of fine log algebraic stacks $([\overline{Y}']/G_{Y'})$,

$M_{[\overline{Y}'/G_{Y'}]} \longrightarrow ([\overline{Y}/G_Y], M_{[\overline{Y}/G_Y]})$, and by the functoriality, the functor (2.56) induces the functor

$$(2.57) \quad \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} \text{Isoc}^{\log}([\overline{Y}/G_Y], M_{[\overline{Y}/G_Y]}) \longrightarrow \varinjlim_{(n,p)=1} \text{Isoc}^{\log}((\overline{X}, Z)^{1/n}, M_{(\overline{X}, Z)^{1/n}}),$$

which is the log version of the functor (2.43).

Next let us consider the version with exponent condition. let us put $\overline{X} \setminus X =: \{z_1, \dots, z_r\}$ and take $\Sigma = \prod_{i=1}^r \Sigma_i \subseteq \mathbb{Z}_p^r$. Then we have the limit

$$\varinjlim_{(n,p)=1} \text{Isoc}^{\log}((\overline{X}, Z)^{1/n}, M_{(\overline{X}, Z)^{1/n}})_{n\Sigma(-ss)}$$

as we have seen in Section 2.3.

On the other hand, For $a_Y : Y \longrightarrow X \in \mathcal{G}_X^t$, let $a_{\overline{Y}} : \overline{Y} \longrightarrow \overline{X}$ be the induced morphism. For $1 \leq i \leq r$, let $e_{\overline{Y},i}$ be the ramification index of $a_{\overline{Y}}$ at a point z in $a_{\overline{Y}}^{-1}(z_i)$ (which is independent of the choice of z) and put $e_{\overline{Y}} := (e_{\overline{Y},i})_{i=1}^r$. Then, we can define the decomposition $\{M_{[\overline{Y}/G_Y],i}\}_{i=1}^r$ as in Example 2.15 and so we can define the category $\text{Isoc}^{\log}([\overline{Y}/G_Y], M_{[\overline{Y}/G_Y]})_{e_{\overline{Y}}\Sigma(-ss)}$ (see Definition 2.9). We also have the category $\text{Isoc}^{\log}(\overline{Y}, M_{\overline{Y}})_{e_{\overline{Y}}\Sigma(-ss)}$, using the decomposition $\{M_{[\overline{Y}/G_Y],i}|_{\overline{Y}}\}_{i=1}^r$ of $M_{\overline{Y}}$. (Note that this decomposition corresponds to the decomposition of the simple normal crossing divisor $(\overline{Y} \setminus Y)_{\text{red}}$ into the subdivisors $\{a_{\overline{Y}}^{-1}(z_i)_{\text{red}}\}_{i=1}^r$.) Let $f : Y' \longrightarrow Y$ be a morphism in \mathcal{G}_X^t and let $\overline{f} : (\overline{Y}', M_{\overline{Y}'}) \longrightarrow (\overline{Y}, M_{\overline{Y}})$ be the induced morphism which is finite Kummer log etale. Let us take $z \in a_{\overline{Y}}^{-1}(z_i)$ and $z' \in \overline{f}^{-1}(z)$, and put $e := e_{\overline{Y}',i}/e_{\overline{Y},i}$. Then we have $\overline{f}^* z = e z'$ etale locally on z and z' . So, by Proposition 1.22, there exists the canonical restriction functor

$$\text{Isoc}^{\log}(\overline{Y}, M_{\overline{Y}})_{e_{\overline{Y}}\Sigma(-ss)} \longrightarrow \text{Isoc}^{\log}(\overline{Y}', M_{\overline{Y}'})_{e_{\overline{Y}'}\Sigma(-ss)}$$

and it induces the canonical restriction functor

$$\text{Isoc}^{\log}([\overline{Y}/G_Y], M_{[\overline{Y}/G_Y]})_{e_{\overline{Y}}\Sigma(-ss)} \longrightarrow \text{Isoc}^{\log}([\overline{Y}'/G_{Y'}], M_{[\overline{Y}'/G_{Y'}]})_{e_{\overline{Y}'}\Sigma(-ss)}.$$

So we have the limit $\varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} \text{Isoc}^{\log}([\overline{Y}/G_Y], M_{[\overline{Y}/G_Y]})_{e_{\overline{Y}}\Sigma(-ss)}$.

In the following, for a Kummer log etale morphism $\varphi : (S, M_S) \longrightarrow (\overline{X}, Z)$ from an fs log scheme (S, M_S) , we regard that the log structure M_S is endowed with the decomposition $\{M_{S,i}\}_{i=1}^r$, where $M_{S,i}$ is the log structure associated to $(\varphi^* z_i)_{\text{red}}$. (Note that S is necessarily a smooth curve and M_S is necessarily equal to the log structure associated to $(\varphi^* Z)_{\text{red}}$.) Then, by Proposition 2.12, the first functor in (2.56) induces the functor

$$(2.58) \quad \text{Isoc}^{\log}([\overline{Y}/G_Y], M_{[\overline{Y}/G_Y]})_{e_{\overline{Y}}\Sigma(-ss)} \longrightarrow \text{Isoc}^{\log}(\overline{Y}_{\bullet}, M_{\overline{Y}_{\bullet}})_{e_{\overline{Y}}\Sigma(-ss)}$$

and the third and the fourth functor in (2.56) induces the functor

$$(2.59) \quad \begin{aligned} \mathrm{Isoc}^{\log}(U_{\bullet\bullet\bullet}, M_{U_{\bullet\bullet\bullet}})_{n\Sigma(-ss)} &\longrightarrow \mathrm{Isoc}^{\log}(\overline{X}_{\bullet\bullet}, M_{\overline{X}_{\bullet\bullet}})_{n\Sigma(-ss)} \\ &\longrightarrow \mathrm{Isoc}((\overline{X}, Z)^{1/n}, M_{(\overline{X}, Z)^{1/n}})_{n\Sigma(-ss)} \end{aligned}$$

because the morphisms $(\overline{Y}_k, M_{\overline{Y}_k}) \longrightarrow ([\overline{Y}/G_Y], M_{[\overline{Y}/G_Y]})$, $(U_{klm}, M_{U_{klm}}) \longrightarrow (\overline{X}_{lm}, M_{\overline{X}_{lm}}) \longrightarrow ((\overline{X}, Z)^{1/n}, M_{(\overline{X}, Z)^{1/n}})$ are strict etale. Let us consider the second functor in (2.56). By definition, the square

$$(2.60) \quad \begin{array}{ccc} (U_{klm}, M_{U_{klm}}) & \longrightarrow & (\overline{X}_{lm}, M_{\overline{X}_{lm}}) \\ \downarrow & & \downarrow \\ (\overline{Y}_k, M_{\overline{Y}_k}) & \longrightarrow & (\overline{X}, Z) \end{array}$$

is Cartesian in the category of fs log schemes. Let us take points in the schemes in (2.60) over z_i

$$\begin{array}{ccc} z'' & \xrightarrow{\quad} & z' \\ \downarrow & & \downarrow \\ z & \xrightarrow{\quad} & z_i. \end{array}$$

Then, etale locally around z, z_i and z' , the diagram $(\overline{Y}_k, M_{\overline{Y}_k}) \longrightarrow (\overline{X}, Z) \longleftarrow (\overline{X}_{lm}, M_{\overline{X}_{lm}})$ admits a chart of the following form:

$$\begin{array}{ccccc} \mathcal{O}_{\overline{Y}_k} & \longleftarrow & \mathcal{O}_{\overline{X}} & \longrightarrow & \mathcal{O}_{\overline{X}_{lm}} \\ \uparrow & & \uparrow & & \uparrow \\ \mathbb{N} & \xleftarrow{e_{\overline{Y},i}} & \mathbb{N} & \xrightarrow{n} & \mathbb{N}. \end{array}$$

Note that, since $(\overline{Y}, M_{\overline{Y}}) \longrightarrow (\overline{X}, M_{\overline{X}})$ is of n -Kummer type by definition of n in the proof of Proposition 2.23, we have $e_{\overline{Y},i} \mid n$. Then we have the isomorphism of monoids

$$(\mathbb{N} \oplus_{e_{\overline{Y},i}, \mathbb{N}, n} \mathbb{N})^{\mathrm{sat}} \xrightarrow{=} \mathbb{N} \oplus \mathbb{Z}/e_{\overline{Y},i}\mathbb{Z}, \quad (1, 0) \mapsto (n/e_{\overline{Y},i}, 1), (0, 1) \mapsto (1, 0),$$

where $(-)^{\mathrm{sat}}$ denotes the saturation. So, etale locally around z and z'' , the left vertical arrow of (2.60) admits a chart

$$\begin{array}{ccc} \mathcal{O}_{\overline{Y}_k} & \longrightarrow & \mathcal{O}_{U_{klm}} \\ \uparrow & & \uparrow \\ \mathbb{N} & \xrightarrow{n/e_{\overline{Y},i}} & \mathbb{N}. \end{array}$$

From this diagram, we see by Proposition 1.22 that the functor $h_{\bullet\bullet\bullet}^*$ in (2.56) induces the functor

$$(2.61) \quad \mathrm{Isoc}^{\log}(\overline{Y}_{\bullet}, M_{\overline{Y}_{\bullet}})_{e_{\overline{Y}}\Sigma(-ss)} \longrightarrow \mathrm{Isoc}^{\log}(U_{\bullet\bullet\bullet}, M_{U_{\bullet\bullet\bullet}})_{n\Sigma(-ss)}.$$

By (2.58), (2.59) and (2.61), we obtain the functor

$$\mathrm{Isoc}^{\log}([\overline{Y}/G_Y], M_{[\overline{Y}/G_Y]})_{e_Y \Sigma(-ss)} \longrightarrow \mathrm{Isoc}^{\log}((\overline{X}, Z)^{1/n}, M_{(\overline{X}, Z)^{1/n}})_{n\Sigma(-ss)}$$

and by functoriality, it induces the functor

$$(2.62) \quad \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} \mathrm{Isoc}^{\log}([\overline{Y}/G_Y], M_{[\overline{Y}/G_Y]})_{e_Y \Sigma(-ss)} \longrightarrow \varinjlim_{(n,p)=1} \mathrm{Isoc}^{\log}((\overline{X}, Z)^{1/n}, M_{(\overline{X}, Z)^{1/n}})_{n\Sigma(-ss)},$$

which is the log version with exponent condition of the functor (2.43).

Keep the assumption that X is a curve. Then we can define the functor

$$\mathrm{Isoc}^{\log}([\overline{Y}/G_Y], M_{\overline{Y}/G_Y})_{(\Sigma(-ss))} \longrightarrow \mathrm{Isoc}^{\dagger}(X, \overline{X})$$

in the same way as (2.19) as the composite

$$\begin{aligned} \mathrm{Isoc}^{\log}([\overline{Y}/G_Y], M_{\overline{Y}/G_Y})_{(\Sigma(-ss))} &\xrightarrow{=} \mathrm{Isoc}^{\log}(\overline{Y}_{\bullet}, M_{\overline{Y}_{\bullet}})_{(\Sigma(-ss))} \\ &\longrightarrow \mathrm{Isoc}^{\dagger}(Y_{\bullet}, \overline{Y}_{\bullet}) \xleftarrow{=} \mathrm{Isoc}^{\dagger}(X, \overline{X}). \end{aligned}$$

Hence we have the functor

$$(2.63) \quad \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} \mathrm{Isoc}^{\log}([\overline{Y}/G_Y], M_{\overline{Y}/G_Y})_{(e_Y \Sigma(-ss))} \longrightarrow \mathrm{Isoc}^{\dagger}(X, \overline{X}).$$

We can check the commutativity of the diagram

$$\begin{array}{ccc} \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} \mathrm{Isoc}^{\log}([\overline{Y}/G_Y], M_{\overline{Y}/G_Y})_{(e_Y \Sigma(-ss))} & \xrightarrow{(2.63)} & \mathrm{Isoc}^{\dagger}(X, \overline{X}) \\ (2.57), (2.62) \downarrow & & \parallel \\ \varinjlim_{(n,p)=1} \mathrm{Isoc}^{\log}((\overline{X}, Z)^{1/n}, M_{(\overline{X}, Z)^{1/n}})_{(n\Sigma(-ss))} & \xrightarrow{(2.54)} & \mathrm{Isoc}^{\dagger}(X, \overline{X}) \end{array}$$

in the same way as that of (2.42), (2.44).

Now we compare the categories $\mathrm{Rep}_{K^{\sigma}}(\pi_1^t(X))$ and $\varinjlim_{(n,p)=1} F\text{-Isoc}((\overline{X}, Z)^{1/n})^{\circ}$:

Theorem 2.26. *The composite*

$$(2.64) \quad \begin{aligned} \mathrm{Rep}_{K^{\sigma}}(\pi_1^t(X)) &\xrightarrow{(2.16)} \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} F\text{-Isoc}([\overline{Y}^{\mathrm{sm}}/G_Y])^{\circ} \\ &\xrightarrow{(2.40)^{\circ}} \varinjlim_{(n,p)=1} F\text{-Isoc}((\overline{X}, Z)^{1/n})^{\circ} \end{aligned}$$

is an equivalence of categories. (In particular, the functor $(2.40)^{\circ}$ is an equivalence.)

The equivalence (2.64) is nothing but (0.9), which is a p -adic version of (0.4).

Proof. Note that a part of the functors (2.49) $^\circ$

$$F\text{-Isoc}(\overline{Y}_{\bullet}^{\text{sm}})^\circ \longrightarrow F\text{-Isoc}(\overline{U}_{\bullet\bullet})^\circ \xleftarrow{=} F\text{-Isoc}(\overline{X}'_{\bullet\bullet}) \xleftarrow{=} F\text{-Isoc}(\overline{X}_{\bullet\bullet})^\circ$$

is rewritten via the equivalence (2.4) in the following way:

$$(2.65) \quad \text{Sm}_{K^\sigma}(\overline{Y}_{\bullet}^{\text{sm}}) \longrightarrow \text{Sm}_{K^\sigma}(\overline{U}_{\bullet\bullet}) \xleftarrow{=} \text{Sm}_{K^\sigma}(\overline{X}'_{\bullet\bullet}) \xleftarrow{=} \text{Sm}_{K^\sigma}(\overline{X}_{\bullet\bullet}).$$

(Here $\overline{Y}_{\bullet}^{\text{sm}}, \overline{U}_{\bullet\bullet}, \overline{X}'_{\bullet\bullet}, \overline{X}_{\bullet\bullet}$ are as in the proof of Proposition 2.23.) Since $\overline{X}_{\bullet\bullet}$, being a part of the data of bisimplicial resolution of $(\overline{X}, Z)^{1/n}$, depends on n , let us denote it by $\overline{X}_{\bullet\bullet}^{(n)}$ in the sequel in this proof. Then (2.65) induces the functor

$$(2.66) \quad \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} \text{Sm}_{K^\sigma}(\overline{Y}_{\bullet}^{\text{sm}}) \longrightarrow \varinjlim_{(n,p)=1} \text{Sm}_{K^\sigma}(\overline{X}_{\bullet\bullet}^{(n)})$$

and using this, we can rewrite the functor (2.64) as the composite

$$(2.67) \quad \text{Rep}_{K^\sigma}(\pi_1^t(X)) \xrightarrow{=} \varinjlim_{Y \rightarrow X \in \mathcal{G}} \text{Sm}_{K^\sigma}(\overline{Y}_{\bullet}^{\text{sm}}) \xrightarrow{(2.66)} \varinjlim_{(n,p)=1} \text{Sm}_{K^\sigma}(\overline{X}_{\bullet\bullet}^{(n)}) \\ \xrightarrow{(2.4),=} \varinjlim_{(n,p)=1} F\text{-Isoc}(\overline{X}_{\bullet\bullet}^{(n)}) \xleftarrow{=} \varinjlim_{(n,p)=1} F\text{-Isoc}((\overline{X}, Z)^{1/n}).$$

To prove the theorem, it suffices to prove that the first line

$$(2.68) \quad \text{Rep}_{K^\sigma}(\pi_1^t(X)) \xrightarrow{=} \varinjlim_{Y \rightarrow X \in \mathcal{G}} \text{Sm}_{K^\sigma}(\overline{Y}_{\bullet}^{\text{sm}}) \xrightarrow{(2.66)} \varinjlim_{(n,p)=1} \text{Sm}_{K^\sigma}(\overline{X}_{\bullet\bullet}^{(n)})$$

of the functor (2.67) is an equivalence. By construction, the composite of the functor (2.68) and the restriction functor

$$(2.69) \quad \varinjlim_{(n,p)=1} \text{Sm}_{K^\sigma}(\overline{X}_{\bullet\bullet}^{(n)}) \longrightarrow \text{Sm}_{K^\sigma}(X_{\bullet\bullet})$$

($X_{\bullet\bullet}$ are as in the proof of Proposition 2.23) is equal to the composite $\text{Rep}_{K^\sigma}(\pi_1^t(X)) \xrightarrow{\subseteq} \text{Rep}_{K^\sigma}(\pi_1(X)) \xrightarrow{=} \text{Sm}_{K^\sigma}(X_{\bullet\bullet})$, which is fully faithful. Also, it is easy to see that (2.69) is faithful. So (2.68) is fully faithful. Also, it is obvious that any object ρ in $\text{Rep}_{K^\sigma}(\pi_1(X))$ which is sent to an object in $\varinjlim_{(n,p)=1} \text{Sm}_{K^\sigma}(\overline{X}_{\bullet\bullet}^{(n)}) \subset \text{Sm}_{K^\sigma}(X_{\bullet\bullet})$ is tamely ramified along Z . So the functor (2.68) is an equivalence, as desired. \square

We have seen above that the functor (2.40) $^\circ$ is an equivalence. So it is natural to ask the following question.

Question 2.27. *Are the functors (2.40), (2.43), (2.57), (2.62) equivalences?*

Here first we prove that the functors (2.40) and (2.43) are equivalences for curves, although we postpone a part of the proof to the next section.

Theorem 2.28. *Let $X \hookrightarrow \bar{X}$ be an open immersion of connected smooth curves such that $\bar{X} \setminus X =: Z$ is a simple normal crossing divisor (= disjoint union of closed points). Then, the functor (2.43)*

$$\varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} \text{Isoc}([\bar{Y}/G_Y]) \longrightarrow \varinjlim_{(n,p)=1} \text{Isoc}((\bar{X}, Z)^{1/n})$$

is an equivalence of categories (hence so is (2.40)).

Before the proof, we introduce one terminology: In the following, a smooth connected curve C over k is called a (g, l) -curve when the smooth compactification \bar{C} of C has genus g and $(\bar{C} \setminus C) \otimes_k \bar{k}$ consists of l points.

Proof. Assume that \bar{X} is a (g, l) -curve and X is a (g, l') -curve ($l \leq l'$). First we prove the theorem in the case $(g, l, l') \neq (0, 0, 1)$. Fix a positive integer n prime to p for the moment, and take a chart $(\bar{X}_0, \{t_i\}_{i=1}^r)$ for (\bar{X}, Z) in the sense of Section 2.3. Let $(\bar{X}_\bullet, M_{\bar{X}_\bullet})$, $(\bar{X}_{\bullet\bullet}, M_{\bar{X}_{\bullet\bullet}})$ be the simplicial semi-resolution, the bisimplicial resolution of $(\bar{X}, Z)^{1/n}$ associated to the chart $(\bar{X}_0, \{t_i\}_{i=1}^r)$, respectively. Let us consider the following claim:

claim 1. Assume $(g, l, l') \neq (0, 0, 1)$. For any n , there exists a finite Kummer log étale Galois morphism $(\bar{Y}, M_{\bar{Y}}) \longrightarrow (\bar{X}, M_{\bar{X}})$ such that if we put $(\tilde{Y}, M_{\tilde{Y}}) := (\bar{Y}, M_{\bar{Y}}) \times_{(\bar{X}, M_{\bar{X}})} (\bar{X}_{00}, M_{\bar{X}_{00}})$ (the fiber product in the category of fs log schemes), the projection $(\tilde{Y}, M_{\tilde{Y}}) \longrightarrow (\bar{Y}, M_{\bar{Y}})$ is strict étale.

First we prove that the claim implies the theorem in the case $(g, l, l') \neq (0, 0, 1)$. Take a positive integer n prime to p , take a finite Kummer log étale Galois morphism $(\bar{Y}, M_{\bar{Y}}) \longrightarrow (\bar{X}, M_{\bar{X}})$ as in the claim and denote its Galois group by G_Y . For $m = 0, 1, 2$, let $(\bar{Y}_m, M_{\bar{Y}_m})$ be the $(m+1)$ -fold fiber product of $(\bar{Y}, M_{\bar{Y}})$ over $(\bar{X}, M_{\bar{X}})$ and for $k, l, m \in \{0, 1, 2\}$, let us put $(\tilde{Y}_{kl}, M_{\tilde{Y}_{kl}}) := (\bar{Y}_k, M_{\bar{Y}_k}) \times_{(\bar{X}, M_{\bar{X}})} (\bar{X}_l, M_{\bar{X}_l})$, $(\tilde{Y}_{klm}, M_{\tilde{Y}_{klm}}) := (\bar{Y}_k, M_{\bar{Y}_k}) \times_{(\bar{X}, M_{\bar{X}})} (\bar{X}_{lm}, M_{\bar{X}_{lm}})$. Let us prove the following claim:

claim 2. With the above notation, we have the following:

- (1) The morphism $(\tilde{Y}_{k\bullet}, M_{\tilde{Y}_{k\bullet}}) \longrightarrow (\bar{Y}_k, M_{\bar{Y}_k})$ induced by the canonical projection is a 2-truncated strict étale Čech hypercovering.
- (2) The morphism $(\tilde{Y}_{kl0}, M_{\tilde{Y}_{kl0}}) \longrightarrow (\tilde{Y}_{kl}, M_{\tilde{Y}_{kl}})$ is strict étale and it induces the 2-truncated strict étale Čech hypercovering $(\tilde{Y}_{kl\bullet}, M_{\tilde{Y}_{kl\bullet}}) \longrightarrow (\tilde{Y}_{kl}, M_{\tilde{Y}_{kl}})$.

Let us prove claim 2 (assuming claim 1). The assertion (1) is easy because the

morphism in question is the base change of the 2-truncated strict etale Čech hypercovering $(\overline{X}_\bullet, M_{\overline{X}_\bullet}) \longrightarrow (\overline{X}, M_{\overline{X}})$ by the morphism $(\overline{Y}_k, M_{\overline{Y}_k}) \longrightarrow (\overline{X}, M_{\overline{X}})$. Let us prove the former assertion of (2). First, by claim 1, the morphism

$$(\tilde{Y}, M_{\tilde{Y}}) = (\overline{Y}, M_{\overline{Y}}) \times_{(\overline{X}, M_{\overline{X}})} (\overline{X}_{00}, M_{\overline{X}_{00}}) \longrightarrow (\overline{Y}, M_{\overline{Y}})$$

is strict etale. By pulling it back by the morphism

$$(\tilde{Y}_{kl}, M_{\tilde{Y}_{kl}}) = (\overline{Y}_k, M_{\overline{Y}_k}) \times_{(\overline{X}, M_{\overline{X}})} (\overline{X}_l, M_{\overline{X}_l}) \longrightarrow (\overline{Y}_k, M_{\overline{Y}_k}) \longrightarrow (\overline{Y}, M_{\overline{Y}}),$$

we see that the morphism

$$(2.70) \quad (\overline{Y}_k, M_{\overline{Y}_k}) \times_{(\overline{X}, M_{\overline{X}})} (\overline{X}_l, M_{\overline{X}_l}) \times_{(\overline{X}, M_{\overline{X}})} (\overline{X}_{00}, M_{\overline{X}_{00}}) \longrightarrow (\tilde{Y}_{kl}, M_{\tilde{Y}_{kl}})$$

is strict etale. Now note that the log structure on $(\overline{X}_l, M_{\overline{X}_l}) \times_{(\overline{X}, M_{\overline{X}})} (\overline{X}_{00}, M_{\overline{X}_{00}})$, being equal to the pull back of the log structure $M_{\overline{X}_{00}}$, is isomorphic to the pull back of the log structure $M_{(\overline{X}, Z)^{1/n}}$ on $(\overline{X}, Z)^{1/n}$ by the etale morphism $\overline{X}_l \times_{\overline{X}} \overline{X}_{00} \longrightarrow \overline{X}_l \times_{\overline{X}} (\overline{X}, Z)^{1/n} \longrightarrow (\overline{X}, Z)^{1/n}$. On the other hand, by definition, the log structure $M_{\overline{X}_{l0}}$ is also isomorphic to the pull back of the log structure $M_{(\overline{X}, Z)^{1/n}}$ on $(\overline{X}, Z)^{1/n}$ by the etale morphism $\overline{X}_{l0} \longrightarrow \overline{X}_l \times_{\overline{X}} (\overline{X}, Z)^{1/n} \longrightarrow (\overline{X}, Z)^{1/n}$. So the canonical morphism

$$(2.71) \quad (\overline{X}_{l0}, M_{\overline{X}_{l0}}) \longrightarrow (\overline{X}_l, M_{\overline{X}_l}) \times_{(\overline{X}, M_{\overline{X}})} (\overline{X}_{00}, M_{\overline{X}_{00}})$$

is strict etale. So the composite

$$\begin{aligned} (\tilde{Y}_{kl0}, M_{\tilde{Y}_{kl0}}) &= (\overline{Y}_k, M_{\overline{Y}_k}) \times_{(\overline{X}, M_{\overline{X}})} (\overline{X}_{l0}, M_{\overline{X}_{l0}}) \\ &\xrightarrow{\text{id} \times (2.71)} (\overline{Y}_k, M_{\overline{Y}_k}) \times_{(\overline{X}, M_{\overline{X}})} (\overline{X}_l, M_{\overline{X}_l}) \times_{(\overline{X}, M_{\overline{X}})} (\overline{X}_{00}, M_{\overline{X}_{00}}) \xrightarrow{(2.70)} (\tilde{Y}_{kl}, M_{\tilde{Y}_{kl}}) \end{aligned}$$

is strict etale, as desired.

Let us prove the latter assertion of (2). Since the morphism $(\tilde{Y}_{kl\bullet}, M_{\tilde{Y}_{kl\bullet}}) \longrightarrow (\tilde{Y}_{kl}, M_{\tilde{Y}_{kl}})$ is the pull back of the 2-truncated log etale Čech hypercovering $(\overline{X}_{l\bullet}, M_{\overline{X}_{l\bullet}}) \longrightarrow (\overline{X}_l, M_{\overline{X}_l})$, it is also a 2-truncated log etale Čech hypercovering. Then, $(\tilde{Y}_{kl0}, M_{\tilde{Y}_{kl0}}) \longrightarrow (\tilde{Y}_{kl}, M_{\tilde{Y}_{kl}})$ is strict etale by the former assertion of (2), we can conclude that it is a 2-truncated strict etale Čech hypercovering. So we have proved the assertion (2).

By claim 2, we can define the functor

$$\begin{aligned} (2.72) \quad \text{Isoc}((\overline{X}, Z)^{1/n}) &\xrightarrow{=} \text{Isoc}(\overline{X}_{\bullet\bullet}) \\ &\longrightarrow \text{Isoc}(\tilde{Y}_{\bullet\bullet\bullet}) \\ &\xrightarrow{=} \text{Isoc}(\tilde{Y}_{\bullet\bullet}) \\ &\xrightarrow{=} \text{Isoc}(\overline{Y}_{\bullet}) = \text{Isoc}([\overline{Y}/G_Y]). \end{aligned}$$

Varying n , we obtain the functor

$$\varinjlim_{(n,p)=1} \text{Isoc}((\overline{X}, Z)^{1/n}) \longrightarrow \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} \text{Isoc}([\overline{Y}/G_Y]),$$

which gives the inverse of (2.43) by construction. So it suffices to prove the claim 1 for the theorem in the case $(g, l, l') \neq (0, 0, 1)$.

Let us prove the claim 1. To prove the claim, we may replace k by its algebraic closure because the scalar extension is ind-étale. So we may assume that k is algebraically closed. First we prove the claim in the case $l' - l \geq 2$. Let us put $\overline{X} \setminus X = \{z_1, \dots, z_r\}$, let I_i be the inertia group at z_i and let $\iota_i : I_i \rightarrow \pi_1(X)$ be the canonical map (defined up to conjugate). Then, to prove the claim, it suffices to show the following: When we are given open normal subgroups $J_i \triangleleft I_i$ ($1 \leq i \leq r$) containing wild inertia subgroup, there exists an open normal subgroup $N \triangleleft \pi_1(X)$ such that, for any $1 \leq i \leq r$, $\iota_i^{-1}(N)$ is contained in J_i and contains wild inertia subgroup. (In fact, we obtain claim 1 if we put J_i 's so that $I_i/J_i \cong \mathbb{Z}/n\mathbb{Z}$ and if we define $(\overline{Y}, M_{\overline{Y}}) \rightarrow (\overline{X}, M_{\overline{X}})$ to be the finite Kummer log étale Galois covering corresponding to N .) Let us consider the following maps

$$I_i \xrightarrow{\psi_i} \pi_1(\text{Spec Frac}(\mathcal{O}_{\overline{X}, z_i}^h)) \xrightarrow{\psi'_i} \pi_1(X) \xrightarrow{\pi} \pi_1(\overline{X} \setminus \{z_i, z_{i+1}\}) (\text{tame at } z_{i+1}),$$

where we put $z_{r+1} := z_1$. (Note that $\iota_i = \psi'_i \circ \psi_i$.) Then, by Katz [19], there exists a morphism $\alpha : \pi_1(\overline{X} \setminus \{z_i, z_{i+1}\}) (\text{tame at } z_{i+1}) \rightarrow \pi_1(\text{Spec Frac}(\mathcal{O}_{\overline{X}, z_i}^h))$ satisfying $\alpha \circ \pi \circ \psi'_i = \text{id}$. Let us take an open normal subgroup $J'_i \triangleleft \pi_1(\text{Spec Frac}(\mathcal{O}_{\overline{X}, z_i}^h))$ with $\psi_i^{-1}(J'_i) = J_i$. (It is possible because the tame inertia group of the henselian field $\text{Frac}(\mathcal{O}_{\overline{X}, z_i}^h)$ is equal to the tame inertia group of its completion.) Then, if we put $N_i := \pi^{-1}\alpha^{-1}(J'_i)$, we have $\iota_i^{-1}(N_i) = J_i$. Moreover, by construction, $\iota_j^{-1}(N_i)$ contains the wild inertia subgroup of I_j even for $j \neq i$. Therefore, if we define N to be the maximal normal subgroup of $\pi_1(X)$ contained in $\bigcap_{i=1}^r N_i$, this N satisfies the required property.

In the case $l' = l$, we have $X = \overline{X}$ and the claim 1 is obviously true because all the log structures appearing in the statement of claim 1 are trivial in this case. So the case $l' - l = 1$ remains unproved. In this case, we have $(g, l) \neq (0, 0)$ because we assumed $(g, l, l') \neq (0, 0, 1)$ for the moment. Then we have a non-trivial finite étale covering $\overline{X}' \rightarrow \overline{X}$, and if we put $X' := X \times_{\overline{X}} \overline{X}'$, we have $|(\overline{X}' \setminus X')(k)| \geq 2$. Then the claim 1 for $X \subseteq \overline{X}$ is reduced to the claim 1 for $X' \subseteq \overline{X}'$ and this is true by the previous argument. So we have proved the claim 1 and so the proof of the theorem is finished when $(g, l, l') \neq (0, 0, 1)$.

In the case $(g, l, l') = (0, 0, 1)$, we have $\overline{X} = \mathbb{P}_k^1$. So the category \mathcal{G}_X^t contains only the trivial covering and we have

$$\varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} \text{Isoc}([\overline{Y}/G_Y]) = \text{Isoc}(\overline{X}) = \text{Isoc}(\mathbb{P}_k^1) = \{\text{constant objects}\} \xrightarrow{=} \text{Vect}_K,$$

where a constant object means a finite direct sum of the structure convergent isocrystal \mathcal{O} , Vect_K means the category of finite dimensional vector spaces over K and the last functor is defined by $\mathcal{E} \mapsto \text{Hom}(\mathcal{O}, \mathcal{E})$. (The third equality follows from [35, 4.4] and the fourth equality follows from the equality $\text{Hom}(\mathcal{O}, \mathcal{O}) = K$.) On the other hand, we shall see later (Proposition 3.8) that the category $\varinjlim_{(n,p)=1} \text{Isoc}((\overline{X}, Z)^{1/n})$ is also naturally equivalent to Vect_K . So we have proved the theorem also in this case modulo Proposition 3.8. \square

Secondly we answer (under certain assumption on Σ) the question for the functors (2.57) and (2.62). (We postpone a part of the proof to the next section.)

Theorem 2.29. *Let $X \hookrightarrow \overline{X}$ be an open immersion of connected smooth curves such that $\overline{X} \setminus X =: Z$ is a simple normal crossing divisor (= disjoint union of closed points). Assume moreover that \overline{X} is a (g, l) -curve and that X is a (g, l') -curve (so $l \leq l'$). Then:*

- (1) *If $(g, l, l') \neq (0, 0, 1)$, the functors (2.57) and (2.62) are equivalences.*
- (2) *If $(g, l, l') = (0, 0, 1)$, the functor (2.57) is not an equivalence.*
- (3) *Assume that $\Sigma \subseteq \mathbb{Z}_p$ is (NRD), (SNLD) and that $(g, l, l') = (0, 0, 1)$. Then the functor (2.62) is an equivalence if and only if $\Sigma \cap \mathbb{Z} = \Sigma \cap \mathbb{Z}_{(p)}$.*

Proof. In this proof, we follow the notation in the proof of Theorem 2.28. First let us prove the assertion (1). In the situation in (1), by claim 2 in the proof of Theorem 2.28, we have the log version of the diagram (2.72)

$$\begin{aligned}
 (2.73) \quad \text{Isoc}^{\log}((\overline{X}, Z)^{1/n}, M_{(\overline{X}, Z)^{1/n}}) &\xrightarrow{=} \text{Isoc}^{\log}(\overline{X}_{\bullet\bullet}, M_{\overline{X}_{\bullet\bullet}}) \\
 &\longrightarrow \text{Isoc}^{\log}(\widetilde{Y}_{\bullet\bullet\bullet}, M_{\widetilde{Y}_{\bullet\bullet\bullet}}) \\
 &\xrightarrow{=} \text{Isoc}^{\log}(\widetilde{Y}_{\bullet\bullet}, M_{\widetilde{Y}_{\bullet\bullet}}) \\
 &\xrightarrow{=} \text{Isoc}^{\log}(\overline{Y}_{\bullet}, M_{\overline{Y}_{\bullet}}) \\
 &= \text{Isoc}^{\log}([\overline{Y}/G_Y], M_{[\overline{Y}/G_Y]})
 \end{aligned}$$

and by definition, it is easy to see that the induced functor

$$\varinjlim_{(n,p)=1} \text{Isoc}^{\log}((\overline{X}, Z)^{1/n}, M_{(\overline{X}, Z)^{1/n}}) \longrightarrow \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} \text{Isoc}^{\log}([\overline{Y}/G_Y], M_{[\overline{Y}/G_Y]})$$

is the inverse of the functor (2.57).

Next let us consider the log version with exponent condition. Let us put $\overline{X} \setminus X = \{z_1, \dots, z_r\}$ and let us take $\Sigma = \prod_{i=1}^r \Sigma_i \subseteq \mathbb{Z}_p^r$. Let us fix a positive integer n prime to p for the moment and choose a morphism $(\overline{Y}, M_{\overline{Y}}) \rightarrow (\overline{X}, M_{\overline{X}})$ as in claim 1. Let us define the ramification index $e_{\overline{Y}} := (e_{\overline{Y}, i})_{i=1}^r$ as in Remark 2.25. Then, in the notation of the proof of claim 1, we have $I_i/\iota_i^{-1}(N) \cong \mathbb{Z}/e_{\overline{Y}, i}\mathbb{Z}$, $I_i/J_i \cong \mathbb{Z}/n\mathbb{Z}$, $\iota^{-1}(N) \subseteq J_i$.

Hence we have $n \mid e_{\overline{Y},i}$. Then, etale locally around any points in the inverse image of z_i , the diagram

$$(\overline{Y}_k, M_{\overline{Y}_k}) \longrightarrow (\overline{X}, M_{\overline{X}}) \longleftarrow (\overline{X}_{lm}, M_{\overline{X}_{lm}})$$

admits a chart of the following form:

$$\begin{array}{ccccc} \overline{Y}_k & \longrightarrow & \overline{X} & \longleftarrow & \overline{X}_{lm} \\ \uparrow & & \uparrow & & \uparrow \\ \mathbb{N} & \xleftarrow{e_{\overline{Y},i}} & \mathbb{N} & \xrightarrow{n} & \mathbb{N}. \end{array}$$

Using this, we see (by the same argument as in Remark 2.25) that the projection

$$(\tilde{Y}_{klm}, M_{\tilde{Y}_{klm}}) = (\overline{Y}_k, M_{\overline{Y}_k}) \times_{(\overline{X}, M_{\overline{X}})} (\overline{X}_{lm}, M_{\overline{X}_{lm}}) \longrightarrow (\overline{X}_{lm}, M_{\overline{X}_{lm}})$$

admits a chart

$$\begin{array}{ccc} \mathcal{O}_{\overline{X}_{lm}} & \longrightarrow & \mathcal{O}_{\tilde{Y}_{klm}} \\ \uparrow & & \uparrow \\ \mathbb{N} & \xrightarrow{e_{\overline{Y},i}/n} & \mathbb{N} \end{array}$$

and from this diagram, we see by Proposition 1.22(2) that the second functor in (2.73) induces the functor

$$\mathrm{Isoc}^{\log}(\overline{X}_{\bullet\bullet}, M_{\overline{X}_{\bullet\bullet}})_{n\Sigma} \longrightarrow \mathrm{Isoc}^{\log}(\tilde{Y}_{\bullet\bullet\bullet}, M_{\bullet\bullet\bullet})_{e_{\overline{Y}}\Sigma}.$$

Since the other functors are induced by strict etale morphisms, we can conclude that (2.73) induces the diagram

$$\begin{aligned} \mathrm{Isoc}^{\log}((\overline{X}, Z)^{1/n}, M_{(\overline{X}, Z)^{1/n}})_{n\Sigma} &\xrightarrow{=} \mathrm{Isoc}^{\log}(\overline{X}_{\bullet\bullet}, M_{\overline{X}_{\bullet\bullet}})_{n\Sigma} \\ &\longrightarrow \mathrm{Isoc}^{\log}(\tilde{Y}_{\bullet\bullet\bullet}, M_{\bullet\bullet\bullet})_{e_{\overline{Y}}\Sigma} \\ &\xrightarrow{=} \mathrm{Isoc}^{\log}(\tilde{Y}_{\bullet\bullet}, M_{\tilde{Y}_{\bullet\bullet}})_{e_{\overline{Y}}\Sigma} \\ &\xrightarrow{=} \mathrm{Isoc}^{\log}(\overline{Y}_{\bullet}, M_{\overline{Y}_{\bullet}})_{e_{\overline{Y}}\Sigma} \\ &= \mathrm{Isoc}^{\log}([\overline{Y}/G_Y], M_{[\overline{Y}/G_Y]})_{e_{\overline{Y}}\Sigma}, \end{aligned}$$

which induces the functor

$$\varinjlim_{(n,p)=1} \mathrm{Isoc}^{\log}((\overline{X}, Z)^{1/n}, M_{(\overline{X}, Z)^{1/n}})_{n\Sigma} \longrightarrow \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} \mathrm{Isoc}^{\log}([\overline{Y}/G_Y], M_{[\overline{Y}/G_Y]})_{e_{\overline{Y}}\Sigma}$$

giving the inverse to the functor (2.62). Hence we have proved the assertion (1).

Next we prove the assertion (2), by showing that the functor (2.57) is not essentially surjective in this case. First, note that, since the category \mathcal{G}_X^t contains only the trivial covering, we have

$$\varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} \mathrm{Isoc}^{\log}([\overline{Y}/G_Y], M_{[\overline{Y}/G_Y]}) = \mathrm{Isoc}^{\log}(\overline{X}, Z).$$

Let us calculate the category on the other hand side. Let us put $Z =: \{z\}$, and take a k -rational point z' of \overline{X} other than z . (It is possible because $\overline{X} \cong \mathbb{P}_k^1$.) Let us put $U := \overline{X} \setminus \{z'\}$, $V := X \cap U$ and for a positive integer n prime to p , let $\varphi^{(n)} : U^{(n)} \rightarrow U$ be the morphism

$$U^{(n)} = \mathbb{A}_k^1 \rightarrow \mathbb{A}_k^1 \cong U$$

induced by $k[t] \rightarrow k[t]; t \mapsto t^n$ and put $V^{(n)} := \varphi^{(n),-1}(V)$. Let M_U (resp. $M_{U^{(n)}}$) be the log structure on U (resp. $U^{(n)}$) associated to $\{z\}$ (resp. $\{0\} \subseteq \mathbb{A}_K^1 = U^{(n)}$). Note that $U^{(n)}$ admits the canonical action (action on the coordinate t) of μ_n over U , and on $[U^{(n)}/\mu_n]$ we have the log structure $M_{[U^{(n)}/\mu_n]}$ induced by $M_{U^{(n)}}$. Then we have

$$\begin{aligned}
(2.74) \quad & \text{Isoc}^{\log}((\overline{X}, Z)^{1/n}, M_{(\overline{X}, Z)^{1/n}}) \\
&= \text{Isoc}^{\log}(X \times_{\overline{X}} ((\overline{X}, Z)^{1/n}, M_{(\overline{X}, Z)^{1/n}})) \times_{\text{Isoc}^{\log}(V \times_{\overline{X}} ((\overline{X}, Z)^{1/n}, M_{(\overline{X}, Z)^{1/n}}))} \\
&\quad \text{Isoc}^{\log}(U \times_{\overline{X}} ((\overline{X}, Z)^{1/n}, M_{(\overline{X}, Z)^{1/n}})) \\
&= \text{Isoc}(X) \times_{\text{Isoc}(V)} \text{Isoc}^{\log}([U^{(n)}/\mu_n], M_{[U^{(n)}/\mu_n]}) \\
&= \text{Isoc}(X) \times_{\text{Isoc}([V^{(n)}/\mu_n])} \text{Isoc}^{\log}([U^{(n)}/\mu_n], M_{[U^{(n)}/\mu_n]}).
\end{aligned}$$

Let us take a diagram (with the square Cartesian)

$$\begin{array}{ccccc}
\mathcal{X} & \xleftarrow{\supset} & \mathcal{V} & \xrightarrow{\subset} & \mathcal{U} \\
(2.75) & & \uparrow & & \uparrow \tilde{\varphi} \\
& & \mathcal{V}^{(n)} & \xrightarrow{\subset} & \mathcal{U}^{(n)}
\end{array}$$

smooth over $\text{Spf } O_K$ lifting

$$\begin{array}{ccccc}
X & \xleftarrow{\supset} & V & \xrightarrow{\subset} & U \\
& & \uparrow & & \uparrow \varphi \\
& & V^{(n)} & \xrightarrow{\subset} & U^{(n)}
\end{array}$$

such that $\mathcal{X}, \mathcal{U}, \mathcal{U}^{(n)}$ are isomorphic to $\widehat{\mathbb{A}}_{O_K}^1$, $\tilde{\varphi}$ is induced by $O_K\{t\} \rightarrow O_K\{t\}; t \mapsto t^n$ and $\mathcal{V}, \mathcal{V}^{(n)} \cong \widehat{\mathbb{G}}_{m, O_K}$. Note that $\mathcal{U}^{(n)}, \mathcal{V}^{(n)}$ admits the canonical action of μ_n which lifts the action of μ_n on $U^{(n)}, V^{(n)}$. Let $\mu_n\text{-LNM}_{\mathcal{U}_K^{(n)}}, \mu_n\text{-LNM}_{\mathcal{V}_K^{(n)}}$ be the category of log- ∇ -modules on $\mathcal{U}_K^{(n)}, \mathcal{V}_K^{(n)}$ with respect to t ($:=$ the coordinate of $\widehat{\mathbb{A}}_{O_K}^1 = \mathcal{U}^{(n)}$) with equivariant μ_n -action. Then we have the canonical fully faithful functors

$$(2.76) \quad \text{Isoc}(X) \rightarrow \text{NM}_{\mathcal{X}_K},$$

$$\begin{aligned}
(2.77) \quad & \text{Isoc}^{\log}([U^{(n)}/\mu_n], M_{[U^{(n)}/\mu_n]}) \\
& \xrightarrow{=} \{\text{objects in } \text{Isoc}^{\log}(U^{(n)}, M_{U^{(n)}}) \text{ with equivariant } \mu_n\text{-action}\} \\
& \rightarrow \mu_n\text{-LNM}_{\mathcal{U}_K^{(n)}},
\end{aligned}$$

$$\begin{aligned}
(2.78) \quad & \text{Isoc}^{\log}([V^{(n)}/\mu_n], M_{[V^{(n)}/\mu_n]}) \\
& \xrightarrow{=} \{\text{objects in } \text{Isoc}^{\log}(V^{(n)}, M_{V^{(n)}}) \text{ with equivariant } \mu_n\text{-action}\} \\
& \longrightarrow \mu_n\text{-LNM}_{\mathcal{V}_K^{(n)}}.
\end{aligned}$$

Let us fix $n \geq 2$ prime to p . We define an object $\mathcal{E} := (\mathcal{E}_0, \mathcal{E}_1, \iota) \in \text{Isoc}^{\log}((\overline{X}, Z)^{1/n}, M_{(\overline{X}, Z)^{1/n}})$ (where $\mathcal{E}_0 \in \text{Isoc}(X)$, $\mathcal{E}_1 \in \text{Isoc}^{\log}([U^{(n)}/\mu], M_{[U^{(n)}/\mu]})$ and ι is the isomorphism between the restriction of \mathcal{E}_1 and \mathcal{E}_0 in the category $\text{Isoc}([V^{(n)}/\mu_n])$) as follows: Let \mathcal{E}_0 be the structure convergent isocrystal on X , which is sent to $(\mathcal{O}_{\mathcal{X}_K}, d)$ by (2.76). Let \mathcal{E}_1 be the unique object in $\text{Isoc}^{\log}([U^{(n)}/\mu], M_{[U^{(n)}/\mu]})$ which is sent by (2.77) to $(t\mathcal{O}_{\mathcal{U}_K^{(n)}}, d|_{t\mathcal{O}_{\mathcal{U}_K^{(n)}}})$ with natural action (the action with $\zeta \cdot t = \zeta t$ for $\zeta \in \mu_n$). (Note that the log- ∇ -module $(t\mathcal{O}_{\mathcal{U}_K^{(n)}}, d|_{t\mathcal{O}_{\mathcal{U}_K^{(n)}}})$ actually comes from a log

convergent isocrystal because the restriction of it to a strict neighborhood of $\mathcal{V}_K^{(n)}$ in $\mathcal{U}_K^{(n)}$ comes from the structure overconvergent isocrystal on $(V^{(n)}, U^{(n)})$. See [21, 6.4.1].) ι is defined to be the isomorphism from the restriction of \mathcal{E}_1 to that of \mathcal{E}_0 defined by $t\mathcal{O}_{\mathcal{U}^{(n)}, K}|_{\mathcal{V}_K^{(n)}} = t\mathcal{O}_{\mathcal{V}_K^{(n)}} \hookrightarrow \mathcal{O}_{\mathcal{V}_K^{(n)}} = \mathcal{O}_{\mathcal{X}_K}|_{\mathcal{V}_K^{(n)}}$. (This is an isomorphism since t is invertible on $\mathcal{V}^{(n)}$.) We denote the induced object in the limit $\varinjlim_{(m,p)=1} \text{Isoc}^{\log}((\overline{X}, Z)^{1/m}, M_{(\overline{X}, Z)^{1/m}})$ also by \mathcal{E} .

We prove that the above \mathcal{E} is not contained in the essential image of (2.57). Assume the contrary. Then $\mathcal{E} \in \text{Isoc}^{\log}((\overline{X}, Z)^{1/m}, M_{(\overline{X}, Z)^{1/m}})$ comes from some object \mathcal{F} in $\text{Isoc}^{\log}(\overline{X}, Z)$ for some m dividing n . Then, by the commutative diagram

$$\begin{array}{ccccc}
\text{Isoc}^{\log}(\overline{X}, Z) & \longrightarrow & \text{Isoc}^{\log}(U, M_U) & \xrightarrow{\subset} & \text{LNM}_{\mathcal{U}_K} \\
(2.57) \downarrow & & & & \downarrow \tilde{\varphi}_K^* \\
\text{Isoc}^{\log}((\overline{X}, Z)^{1/m}, M_{(\overline{X}, Z)^{1/m}}) & \longrightarrow & \text{Isoc}^{\log}([U^{(m)}/\mu_n], M_{[U^{(m)}/\mu_n]}) & \xrightarrow{\subset} & \mu_n\text{-LNM}_{\mathcal{U}_K^{(m)}}
\end{array}$$

(where $\tilde{\varphi}_K^*$ is the pull back by the morphism $\tilde{\varphi}_K : \mathcal{U}_K^{(m)} \rightarrow \mathcal{U}$ induced by $\tilde{\varphi}$), there exists a locally free module of finite rank F on \mathcal{U}_K and a μ_m -equivariant isomorphism $\tilde{\varphi}_K^* F \xrightarrow{=} t^{m/n} \mathcal{O}_{\mathcal{U}_K^{(m)}}$. (Here t is the coordinate of $\mathcal{U}_K^{(m)}$.) But this is impossible since $t^{m/n} \mathcal{O}_{\mathcal{U}_K^{(m)}}$ is not generated by μ_m -invariant sections. So we have a contradiction and so (2.57) is not essentially surjective in this case. Hence we have proved the assertion (2).

Let us prove the assertion (3). As in (2), we see that $\varinjlim_{Y \rightarrow X \in \mathcal{G}_X^k} \text{Isoc}^{\log}([\overline{Y}/G_Y], M_{[\overline{Y}/G_Y]})_{e_{\overline{Y}} \Sigma(-\text{ss})} = \text{Isoc}^{\log}(\overline{X}, Z)_{\Sigma(-\text{ss})}$. For a ring A , Let $\mathbf{X}_A \hookrightarrow \overline{\mathbf{X}}_A \hookleftarrow \mathbf{Z}_A$ be the diagram $\mathbb{A}_A^1 \hookrightarrow \mathbb{P}_A^1 \hookleftarrow \{\infty\}$ over $\text{Spec } A$. When $A = O_K$, this is a lift of the diagram $X \hookrightarrow \overline{X} \hookleftarrow Z$. We denote the p -adic completion of this diagram in the case $A = O_K$ by $\mathcal{X} \hookrightarrow \overline{\mathcal{X}} \hookleftarrow \mathcal{Z}$. Note that we have the canonical fully faithful functor

$$(2.79) \quad \text{Isoc}^{\log}(\overline{X}, Z)_{\Sigma(-\text{ss})} \hookrightarrow \text{LNM}_{(\overline{\mathcal{X}}, \mathcal{Z}), \Sigma} \cong \text{LNM}_{(\overline{\mathbf{X}}_K, \mathbf{Z}_K), \Sigma},$$

where, for a field A , $\text{LNM}_{(\overline{\mathbf{X}}_A, \mathbf{Z}_A), \Sigma}$ denotes the category whose object is a locally free module E of finite rank over $\overline{\mathbf{X}}_A$ endowed with an integrable log connection

$\nabla : E \longrightarrow E \otimes_{\mathcal{O}_{\overline{\mathbf{X}}_A}} \Omega_{\overline{\mathbf{X}}_A/A}^1(\log \mathbf{Z}_A)$ relative to A whose exponents along \mathbf{Z}_A are contained in Σ in algebraic sense. (Note that the latter equivalence in (2.79) follows from GAGA theorem.) Note that, since Σ is (NRD), we have $|\Sigma \cap \mathbb{Z}| \leq 1$. We prove the following claim:

claim. When $\Sigma \cap \mathbb{Z}$ is empty, the category $\mathrm{LNM}_{(\overline{\mathbf{X}}_K, \mathbf{Z}_K), \Sigma}$ is empty. If $\Sigma \cap \mathbb{Z} = \{N\}$ is nonempty, any object in $\mathrm{LNM}_{(\overline{\mathbf{X}}_K, \mathbf{Z}_K), \Sigma}$ is a finite direct sum of the object $(\mathcal{O}_{\overline{\mathbf{X}}_K}(-N\mathbf{Z}_K), d)$.

Let us take $(E, \nabla) \in \mathrm{LNM}_{(\overline{\mathbf{X}}_K, \mathbf{Z}_K), \Sigma}$. Then there exists a countable subfield $K_0 \subseteq K$ and $(E_0, \nabla_0) \in \mathrm{LNM}_{(\overline{\mathbf{X}}_{K_0}, \mathbf{Z}_{K_0})}$ such that $(E, \nabla) = (E_0 \otimes_{K_0} K, \nabla_0 \otimes_{K_0} K)$. Then take an inclusion $K_0 \subseteq \mathbb{C}$. Then we obtain $(E_0 \otimes_{K_0} \mathbb{C}, \nabla_0 \otimes_{K_0} \mathbb{C}) \in \mathrm{LNM}_{(\overline{\mathbf{X}}_{\mathbb{C}}, \mathbf{Z}_{\mathbb{C}}), \Sigma}$ such that the restriction of it to $\mathbf{X}_{\mathbb{C}}^{\mathrm{an}} = \mathbb{A}_{\mathbb{C}}^{1, \mathrm{an}}$ is trivial since $\pi_1(\mathbb{A}_{\mathbb{C}}^{1, \mathrm{an}})$ is trivial. If $\Sigma \cap \mathbb{Z}$ is empty, there does not exist such $(E_0 \otimes_{K_0} \mathbb{C}, \nabla_0 \otimes_{K_0} \mathbb{C})$ by monodromy reason. So the category $\mathrm{LNM}_{(\overline{\mathbf{X}}_K, \mathbf{Z}_K), \Sigma}$ is empty in this case. If $\Sigma \cap \mathbb{Z} = \{N\}$, such $(E_0 \otimes_{K_0} \mathbb{C}, \nabla_0 \otimes_{K_0} \mathbb{C})$ is necessarily isomorphic to a finite direct sum of $(\mathcal{O}_{\overline{\mathbf{X}}_{\mathbb{C}}}(-N\mathbf{Z}_{\mathbb{C}}), d)$ by [10, II 5.4]. So we have

$$\begin{aligned} & \dim_{K_0} \mathrm{Hom}((\mathcal{O}_{\overline{\mathbf{X}}_{K_0}}(-N\mathbf{Z}_{K_0}), d), (E_0, \nabla_0)) \\ &= \dim_{\mathbb{C}} \mathrm{Hom}((\mathcal{O}_{\overline{\mathbf{X}}_{\mathbb{C}}}(-N\mathbf{Z}_{\mathbb{C}}), d), (E_0 \otimes_{K_0} \mathbb{C}, \nabla_0 \otimes_{K_0} \mathbb{C})) = \mathrm{rk} E_0 \end{aligned}$$

and hence (E_0, ∇_0) is isomorphic to a finite direct sum of $(\mathcal{O}_{\overline{\mathbf{X}}_{K_0}}(-N\mathbf{Z}_{K_0}), d)$. Therefore (E, ∇) is isomorphic to a finite direct sum of $(\mathcal{O}_{\overline{\mathbf{X}}_K}(-N\mathbf{Z}_K), d)$, as desired. So the claim is proved.

By claim, $\mathrm{Isoc}(\overline{X}, Z)_{\Sigma(-\mathrm{ss})}$ is empty if $\Sigma \cap \mathbb{Z}$ is empty. Let us consider the case $\Sigma \cap \mathbb{Z} = \{N\}$. In this case, the object $(\mathcal{O}_{\overline{\mathbf{X}}_K}(-N\mathbf{Z}_K), d)$ comes from an object in $\mathrm{Isoc}^{\log}(\overline{X}, Z)_{\Sigma(-\mathrm{ss})}$ since the restriction of it to a strict neighborhood of \mathcal{X}_K in $\overline{\mathcal{X}}_K$ comes from the structure overconvergent isocrystal on (X, \overline{X}) . So, in this case, (2.79) induces the equivalence

$$\mathrm{Isoc}^{\log}(\overline{X}, Z)_{\Sigma(-\mathrm{ss})} \xrightarrow{=} \mathrm{LNM}_{(\overline{\mathcal{X}}, Z), \Sigma} \cong \mathrm{LNM}_{(\mathbf{X}_K, \mathbf{Z}_K), \Sigma} \xrightarrow{=} \mathrm{Vect}_K,$$

where the last functor is defined by $(E, \nabla) \mapsto \mathrm{Hom}((\mathcal{O}_{\overline{\mathbf{X}}_K}(-N\mathbf{Z}_K), d), (E, \nabla))$.

On the other hand, we will prove later (Proposition 3.8) that the category $\varinjlim_{(n,p)=1} \mathrm{Isoc}^{\log}((\overline{X}, Z)^{1/n}, M_{(\overline{X}, Z)^{1/n}})$ is empty when $\Sigma \cap \mathbb{Z}_{(p)}$ is empty and equivalent to Vect_K in compatible way as the above equivalence when $\Sigma \cap \mathbb{Z}_{(p)}$ consists of one element. (Note that we have $|\Sigma \cap \mathbb{Z}_{(p)}| \leq 1$ because Σ is (NRD).) So, in the situation of (3), the functor (2.62) is an equivalence if and only if $\Sigma \cap \mathbb{Z} = \Sigma \cap \mathbb{Z}_{(p)}$. So we have finished the proof of the theorem modulo Proposition 3.8. \square

3 Parabolic log convergent isocrystals

Let $X \hookrightarrow \overline{X}$ be an open immersion of smooth k -varieties such that $Z := \overline{X} \setminus X$ is a simple normal crossing divisor and let $Z = \bigcup_{i=1}^r Z_i$ be the decomposition of

Z into irreducible components. We regard $\{Z_i\}_{i=1}^r$ as the fixed decomposition of Z in the sense of Definition 1.19. In this section, we introduce the category of (semisimply adjusted) parabolic (unit-root) log convergent (F -)isocrystals and prove the equivalence (0.10). In the course of the proof, we prove the equivalence of the variants of right hand sides of (0.9) and (0.10) without Frobenius structures, with log structures and with exponent conditions.

Before the definition of parabolic log convergent isocrystals on (\overline{X}, Z) , first we prove the existence of certain parabolic objects in $\text{Isoc}^{\log}(\overline{X}, Z)$.

Proposition 3.1. *Let $X, \overline{X}, Z = \bigcup_{i=1}^r Z_i$ be as above. There exists a unique inductive system $(\mathcal{O}(\sum_i \alpha_i Z_i))_{\alpha=(\alpha_i) \in \mathbb{Z}^r}$ of objects in $\text{Isoc}^{\log}(\overline{X}, Z)$ (we denote the transition map by $\iota_{\alpha\beta}^0 : \mathcal{O}(\sum_i \alpha_i Z_i) \rightarrow \mathcal{O}(\sum_i \beta_i Z_i)$ for $\alpha = (\alpha_i), \beta = (\beta_i) \in \mathbb{Z}^r$ with $\alpha_i \leq \beta_i (\forall i)$) satisfying the following conditions:*

- (1) $\mathcal{O}(\sum_i \alpha_i Z_i)$ has exponents in $\{-\alpha\} = \prod_{i=1}^r \{-\alpha_i\}$ with semisimple residues.
- (2) The restriction $((j^\dagger \mathcal{O}(\sum_i \alpha_i Z_i))_\alpha, (j^\dagger \iota_{\alpha\beta}^0)_{\alpha,\beta})$ of $((\mathcal{O}(\sum_i \alpha_i Z_i))_\alpha, (\iota_{\alpha\beta}^0)_{\alpha,\beta})$ to an inductive system in $\text{Isoc}^\dagger(X, \overline{X})$ is equal to the constant object $((j^\dagger \mathcal{O}), (\text{id}))$, where $j^\dagger \mathcal{O}$ denotes the structure overconvergent isocrystal on (X, \overline{X}) .

Moreover, it has the following property.

- (3) For any open subscheme $\overline{U} \hookrightarrow \overline{X}$ and a charted standard small frame $((U, \overline{U}, \overline{\mathcal{X}}, i, j), t_1, \dots, t_r)$ enclosing $(U := X \cap \overline{U}, \overline{U})$ with $\mathcal{Z} = \bigcup_{i=1}^r \mathcal{Z}_i$ the lift of $Z \cap \overline{U}$, the inductive system of log- ∇ -module $(E_\alpha, \nabla_\alpha)_\alpha$ on $(\overline{\mathcal{X}}, \mathcal{Z})$ induced by $(\mathcal{O}(\sum_i \alpha_i Z_i))_\alpha$ has the form

$$(3.1) \quad (E_\alpha, \nabla_\alpha) = (\mathcal{O}_{\overline{X}_K}(\sum_i \alpha_i \mathcal{Z}_{i,K}), d),$$

with $\iota_{\alpha\beta}^0$ equal to the canonical inclusion.

Proof. Let $\tau' : (\mathbb{Z}_p/\mathbb{Z}) \setminus \{\overline{0}\} \rightarrow \mathbb{Z}_p$ (where $\overline{0}$ is the class of 0 in \mathbb{Z}_p/\mathbb{Z}) be any section of the projection $\mathbb{Z}_p \rightarrow \mathbb{Z}_p/\mathbb{Z}$, $\tau_n : \mathbb{Z}_p/\mathbb{Z} \rightarrow \mathbb{Z}_p$ be the section of the projection extending τ' with $\tau(\overline{0}) = -n$ and for $\alpha \in \mathbb{Z}^r$, let us put $\tau_\alpha := \prod_{i=1}^r \tau_{\alpha_i}$. Then we have the equivalence of categories

$$(3.2) \quad j^\dagger : \text{Isoc}^{\log}(\overline{X}, Z)_{\tau_\alpha(\mathbf{0})\text{-ss}} \xrightarrow{\cong} \text{Isoc}^\dagger(X, \overline{X})_{\mathbf{0}\text{-ss}}$$

by Theorem 1.17. So there exists a unique object $\mathcal{O}(\sum_i \alpha_i Z_i)$ in $\text{Isoc}^{\log}(\overline{X}, Z)_{\tau_\alpha(\mathbf{0})\text{-ss}} = \text{Isoc}^{\log}(\overline{X}, Z)_{\{-\alpha\}\text{-ss}}$ with $j^\dagger(\mathcal{O}(\sum_i \alpha_i Z_i)) = j^\dagger \mathcal{O}$. Moreover, by Proposition 1.18, we have a unique morphism $\iota_{\alpha\beta}^0 : \mathcal{O}(\sum_i \alpha_i Z_i) \rightarrow \mathcal{O}(\sum_i \beta_i Z_i)$ for any $\alpha = (\alpha_i), \beta = (\beta_i) \in \mathbb{Z}^r$ with $\alpha_i \leq \beta_i (\forall i)$ satisfying $j^\dagger \iota_{\alpha\beta}^0 = \text{id}_{j^\dagger \mathcal{O}}$. Then the resulting inductive system $(\mathcal{O}(\sum_i \alpha_i Z_i))_\alpha$ satisfies the conditions (1), (2). The uniqueness is also clear from the construction.

Let us prove that $(\mathcal{O}(\sum_i \alpha_i Z_i))_\alpha$ satisfies the condition (3). In the situation of (3), the equivalence (3.2) holds also for (U, \overline{U}) . Also, note that the log- ∇ -module (3.1) on $(\overline{\mathcal{X}}_K, \mathcal{Z}_K)$ is restricted to the trivial ∇ -module on a strict neighborhood of $(\overline{\mathcal{X}} \setminus \mathcal{Z})_K$ in $\overline{\mathcal{X}}_K$. Hence it defines an object in $\text{Isoc}^{\log}(\overline{U}, Z \cap \overline{U})$ by [21, 6.4.1] and it is easy to see that it has exponents in $\tau_\alpha(\mathbf{0}) = \{-\alpha\}$ with semisimple residues (with respect to the decomposition $\{Z_i \cap \overline{U}\}_i$ of $Z \cap \overline{U}$). Hence (3.1) defines an object in $\text{Isoc}^{\log}(U, \overline{U})_{\tau_\alpha(\mathbf{0})\text{-ss}}$. Moreover, the transition maps defined in (3) give morphisms in $\text{Isoc}^{\log}(U, \overline{U})$ which reduce to the identity in $\text{Isoc}^\dagger(U, \overline{U})$. So, by the uniqueness in the construction of $(\mathcal{O}(\sum_i \alpha_i Z_i))_\alpha$ (this remains true even when we replace (X, \overline{X}) by (U, \overline{U})), we see that the inductive system of log- ∇ -modules given in (3) is induced from $(\mathcal{O}(\sum_i \alpha_i Z_i))_\alpha$. So the property (3) is also proved. \square

For $\mathcal{E} \in \text{Isoc}^{\log}(\overline{X}, Z)$ and $\alpha = (\alpha_i)_i \in \mathbb{Z}^r$, we define $\mathcal{E}(\sum_i \alpha_i Z_i)$ by $\mathcal{E}(\sum_i \alpha_i Z_i) := \mathcal{E} \otimes \mathcal{O}(\sum_i \alpha_i Z_i)$. We define the notion of parabolic log convergent isocrystals on (\overline{X}, Z) as follows:

Definition 3.2. *Let $X \hookrightarrow \overline{X}$, $Z = \bigcup_{i=1}^r Z_i$ be as above. Then a parabolic log convergent isocrystal on (\overline{X}, Z) is an inductive system $(\mathcal{E}_\alpha)_{\alpha \in \mathbb{Z}_{(p)}^r}$ of objects in $\text{Isoc}^{\log}(\overline{X}, Z)$ (we denote the transition map by $\iota_{\alpha\beta} : \mathcal{E}_\alpha \longrightarrow \mathcal{E}_\beta$ for $\alpha = (\alpha_i), \beta = (\beta_i) \in \mathbb{Z}_{(p)}^r$ with $\alpha_i \leq \beta_i$ ($\forall i$)) satisfying the following conditions:*

- (1) *For any $1 \leq i \leq r$, there is an isomorphism as inductive systems*

$$((\mathcal{E}_{\alpha+e_i})_\alpha, (\iota_{\alpha+\alpha+e_i, \beta+\alpha+e_i})_{\alpha, \beta}) \cong ((\mathcal{E}_\alpha(Z_i))_\alpha, (\iota_{\alpha\beta} \otimes \text{id})_{\alpha, \beta})$$

via which the morphism $(\iota_{\alpha, \alpha+e_i})_\alpha : (\mathcal{E}_\alpha)_\alpha \longrightarrow (\mathcal{E}_{\alpha+e_i})_\alpha$ is identified with the morphism $(\text{id} \otimes \iota_{\alpha, \alpha+e_i}^0)_\alpha : (\mathcal{E}_\alpha)_\alpha \longrightarrow (\mathcal{E}_\alpha(Z_i))_\alpha$.

- (2) *There exists a positive integer n prime to p satisfying the following condition: For any $\alpha = (\alpha_i)_i$, $\iota_{\alpha'\alpha}$ is an isomorphism if we put $\alpha' = ([n\alpha_i]/n)_i$.*

We denote by $\text{Par-Isoc}^{\log}(\overline{X}, Z)$ the category of parabolic log convergent isocrystals on (\overline{X}, Z) .

For a parabolic log convergent isocrystal $(\mathcal{E}_\alpha)_\alpha$ on (\overline{X}, Z) , the transition map $\iota_{\alpha\beta} : \mathcal{E}_\alpha \longrightarrow \mathcal{E}_\beta$ is always injective, because we have the diagram

$$\mathcal{E}_\alpha \xrightarrow{\iota_{\alpha\beta}} \mathcal{E}_\beta \xrightarrow{\iota_{\beta, \alpha+N}} \mathcal{E}_{\alpha+N} \cong \mathcal{E}(\sum_i N_i Z_i)$$

for some $N = (N_i)_i \in \mathbb{Z}^r$ and the composite is injective. Then, applying j^\dagger to the above diagram and noting that $j^\dagger \mathcal{E}_\alpha \longrightarrow j^\dagger \mathcal{E}(\sum_i N_i Z_i)$ is an isomorphism, we see that the inductive system $(j^\dagger \mathcal{E}_\alpha)_\alpha$ of objects in $\text{Isoc}^\dagger(X, \overline{X})$ is constant, that is, $j^\dagger \mathcal{E}_\alpha \in \text{Isoc}^\dagger(X, \overline{X})$ is independent of α .

Next we give a definition of parabolic (unit-root) log convergent F -isocrystals.

Definition 3.3. A parabolic log convergent F -isocrystal on (\overline{X}, Z) is a parabolic log convergent isocrystal $(\mathcal{E}_\alpha)_{\alpha \in \mathbb{Z}_{(p)}^r}$ endowed with an isomorphism $\Psi : \varinjlim (F^* \mathcal{E}_\alpha)_{\alpha \in \mathbb{Z}_{(p)}^r} \xrightarrow{=} \varinjlim (\mathcal{E}_\alpha)_{\alpha \in \mathbb{Z}_{(p)}^r}$ as ind-objects. It is called unit-root if the object $(j^\dagger \mathcal{E}_\alpha, j^\dagger \Psi)$ in $F\text{-Isoc}^\dagger(X, \overline{X})$ induced by $((\mathcal{E}_\alpha)_\alpha, \Psi)$ is unit-root. A morphism $f : ((\mathcal{E}_\alpha)_\alpha, \Psi) \longrightarrow ((\mathcal{E}'_\alpha)_\alpha, \Psi')$ between parabolic log convergent F -isocrystals is defined to be a map of inductive system of log convergent isocrystals $(f_\alpha)_\alpha : (\mathcal{E}_\alpha)_\alpha \longrightarrow (\mathcal{E}'_\alpha)_\alpha$ such that $(\varinjlim_\alpha f_\alpha) \circ \Psi = \Psi' \circ (\varinjlim_\alpha f_\alpha)$ as morphism of ind-objects.

We define the notion of ‘(semisimple) Σ -adjustedness’ for a parabolic log convergent isocrystals as follows:

Definition 3.4. Let $\Sigma = \prod_{i=1}^r \Sigma_i$ be a subset of \mathbb{Z}_p^r which is (NRD) and (SNLD). Then a parabolic log convergent isocrystal $\mathcal{E} := (\mathcal{E}_\alpha)_\alpha$ is called Σ -adjusted (resp. semisimply Σ -adjusted) if, for any $\alpha = (\alpha_i)_i \in \mathbb{Z}_{(p)}^r$, \mathcal{E}_α has exponents in $\prod_{i=1}^r (\Sigma_i + ([-\alpha_i, -\alpha_i + 1] \cap \mathbb{Z}_{(p)}))$. (resp. \mathcal{E}_α has exponents in $\prod_{i=1}^r (\Sigma_i + ([-\alpha_i, -\alpha_i + 1] \cap \mathbb{Z}_{(p)}))$ with semisimple residues.) When $\Sigma = \mathbf{0}$, we call it simply by adjusted (resp. semisimply adjusted). A parabolic log convergent F -isocrystal $((\mathcal{E}_\alpha)_\alpha, \Psi)$ is called adjusted (resp. semisimply adjusted) if so is $(\mathcal{E}_\alpha)_\alpha$.

We denote the category of Σ -adjusted (resp. semisimply Σ -adjusted) parabolic log convergent isocrystals on (\overline{X}, Z) by $\text{Par-Isoc}^{\log}(\overline{X}, Z)_\Sigma$ (resp. $\text{Par-Isoc}^{\log}(\overline{X}, Z)_{\Sigma\text{-ss}}$). Also, we denote the category of adjusted (resp. semisimply adjusted) parabolic log convergent F -isocrystals on (\overline{X}, Z) by $\text{Par-F-Isoc}^{\log}(\overline{X}, Z)_\mathbf{0}$ (resp. $\text{Par-F-Isoc}^{\log}(\overline{X}, Z)_{\mathbf{0}\text{-ss}}$) and the category of semisimply adjusted parabolic unit-root log convergent F -isocrystals on (\overline{X}, Z) by $\text{Par-F-Isoc}^{\log}(\overline{X}, Z)_{\mathbf{0}\text{-ss}}^\circ$.

Remark 3.5. Let $X \hookrightarrow \overline{X}, Z = \bigcup_{i=1}^r Z_i$ be as above. Let Z_{sing} be the set of singular points of Z and assume given a subset $\Sigma = \prod_{i=1}^r \Sigma_i$ of \mathbb{Z}_p^r which is (NRD) and (SNLD). In this remark, we prove that an object $\mathcal{E} := ((\mathcal{E}_\alpha)_\alpha, (\iota_{\alpha\beta})_{\alpha,\beta})$ in $\text{Par-Isoc}^{\log}(\overline{X}, Z)$ is contained in $\text{Par-Isoc}^{\log}(\overline{X}, Z)_{\Sigma\text{-ss}}$ if and only if it satisfies the following condition (**):

(**) For any $1 \leq i \leq r$, for any open subscheme $\overline{U} \subseteq \overline{X} \setminus Z_{\text{sing}}$ containing the generic point of Z_i and any charted smooth standard small frame with generic point $((U, \overline{U}, \overline{\mathcal{X}}), t, L)$ enclosing (U, \overline{U}) (where $U := X \cap \overline{U}$), the inductive system of log- ∇ -modules $(E_{\mathcal{E}, L, \alpha}, \nabla_{\mathcal{E}, L, \alpha})$ on $A_L^1[0, 1)$ induced by \mathcal{E} has the form

$$(3.3) \quad (E_{\mathcal{E}, L, \alpha}, \nabla_{\mathcal{E}, L, \alpha}) = \bigoplus_{j=1}^{\mu} (\mathcal{O}_{A_L^1[0, 1)}, d + (\gamma_j + \lfloor b_j \rfloor - \alpha_i) \text{dlog } t)$$

for some $\mu \in \mathbb{N}, \gamma_j \in \Sigma_i, b_j \in [0, 1) \cap \mathbb{Z}_{(p)}$ ($1 \leq j \leq \mu$) with $\iota_{\alpha\beta}$ equal to the multiplication by $t^{\lfloor b_j \rfloor - \alpha_i - \lfloor b_j \rfloor - \beta_i}$, where, for $a, b \in \mathbb{Z}_{(p)}$, $\lfloor b \rfloor_a$ denotes the unique element in $[a, a + 1) \cap (b + \mathbb{Z})$.

The ‘if’ part is easy because, if we have the equality (3.3), \mathcal{E}_α has exponents in

$\prod_{i=1}^r (\Sigma_i + ([-\alpha_i, -\alpha_i + 1] \cap \mathbb{Z}_{(p)}))$ with semisimple residues by Lemma 1.15(2). Let us prove the ‘only if’ part. By the argument in the proof of Theorem 1.17, we see that $(E_{\mathcal{E}, L, 0}, \nabla_{\mathcal{E}, L, 0})$ is $(\Sigma_i + ([0, 1] \cap \mathbb{Z}_{(p)}))$ -semisimple. So there exists some $\mu \in \mathbb{N}$, $\gamma_j \in \Sigma_i$, $b_j \in [0, 1] \cap \mathbb{Z}_{(p)}$ ($1 \leq j \leq \mu$) such that the equality (3.3) holds for $\alpha = 0$. Let us note now that the inductive system $(\bigoplus_{j=1}^{\mu} (\mathcal{O}_{A_L^1[0,1]}, d + (\gamma_j + \lfloor b_j \rfloor_{-\alpha_i}) \text{dlog } t))_{\alpha}$ above has the following properties:

- (1) $\bigoplus_{j=1}^{\mu} (\mathcal{O}_{A_L^1[0,1]}, d + (\gamma_j + \lfloor b_j \rfloor_{-\alpha_i}) \text{dlog } t)$ is $(\Sigma_i + ([-\alpha_i, -\alpha_i + 1] \cap \mathbb{Z}_{(p)}))$ -semisimple.
- (2) Transition maps are isomorphism on $A_L^1[\lambda, 1)$ for some $\lambda \in (0, 1) \cap \Gamma^*$.

Note also that the inductive system $(E_{\mathcal{E}, L, \alpha}, \nabla_{\mathcal{E}, L, \alpha})_{\alpha}$ also has the same properties. So, by Proposition 1.6, the equality (3.3) for $\alpha = 0$ extends to the isomorphism (3.3) as inductive systems. So we have proved the desired claim.

Note that, by the argument after Definition 3.2, we have the functor

$$(3.4) \quad \text{Par-Isoc}^{\log}(\overline{X}, Z)_{\Sigma(-\text{ss})} \xrightarrow{\subset} \text{Par-Isoc}^{\log}(\overline{X}, Z) \longrightarrow \text{Isoc}^{\dagger}(X, \overline{X})$$

defined by $(\mathcal{E}_{\alpha})_{\alpha} \mapsto j^{\dagger} \mathcal{E}_{\alpha}$. Then the key result in this section is given as follows:

Theorem 3.6. *Let $X, \overline{X}, Z = \bigcup_{i=1}^r Z_i$ be as above and let $\Sigma = \prod_{i=1}^r \Sigma_i$ be a subset of \mathbb{Z}_p^r which is (NRD) and (SNLD). Then there exists the canonical equivalence of categories*

$$(3.5) \quad \varinjlim_{(n,p)=1} \text{Isoc}^{\log}((\overline{X}, Z)^{1/n}, M_{(\overline{X}, Z)^{1/n}})_{n\Sigma(-\text{ss})} \xrightarrow{=} \text{Par-Isoc}^{\log}(\overline{X}, Z)_{\Sigma(-\text{ss})}$$

which makes the following diagram commutative:

$$(3.6) \quad \begin{array}{ccc} \varinjlim_{(n,p)=1} \text{Isoc}^{\log}((\overline{X}, Z)^{1/n}, M_{(\overline{X}, Z)^{1/n}})_{n\Sigma(-\text{ss})} & \xrightarrow{(3.5)} & \text{Par-Isoc}^{\log}(\overline{X}, Z)_{\Sigma(-\text{ss})} \\ \downarrow (2.55) & & \downarrow (3.4) \\ \text{Isoc}^{\dagger}(X, \overline{X}) & \xlongequal{\quad} & \text{Isoc}^{\dagger}(X, \overline{X}). \end{array}$$

Proof. First we prove that the functor (2.55) induces the equivalence of categories

$$(3.7) \quad \varinjlim_{(n,p)=1} \text{Isoc}^{\log}((\overline{X}, Z)^{1/n}, M_{(\overline{X}, Z)^{1/n}})_{n\Sigma(-\text{ss})} \xrightarrow{=} \text{Isoc}^{\dagger}(X, \overline{X})_{\overline{\Sigma}(-\text{ss})},$$

where $\overline{\Sigma} := \text{Im}(\Sigma + \mathbb{Z}_{(p)}^r \hookrightarrow \mathbb{Z}_p^r \longrightarrow \mathbb{Z}_p^r / \mathbb{Z}^r)$. To see this, it suffices to prove that the functor (2.54) induces the equivalence

$$(3.8) \quad \text{Isoc}^{\log}((\overline{X}, Z)^{1/n}, M_{(\overline{X}, Z)^{1/n}})_{n\Sigma(-\text{ss})} \xrightarrow{=} \text{Isoc}^{\dagger}(X, \overline{X})_{\overline{\Sigma}_n(-\text{ss})},$$

where $\overline{\Sigma}_n := \text{Im}(\Sigma + (\frac{1}{n}\mathbb{Z})^r \hookrightarrow \mathbb{Z}_p^r \longrightarrow \mathbb{Z}_p^r / \mathbb{Z}^r)$. We have already seen in Remark 2.24 that the functor

$$(3.9) \quad \text{Isoc}^{\log}((\overline{X}, Z)^{1/n}, M_{(\overline{X}, Z)^{1/n}})_{n\Sigma(-\text{ss})} \longrightarrow \text{Isoc}^{\dagger}(X, \overline{X})$$

defined in (2.54) is fully faithful. So it suffices to prove that essential image of this functor is equal to $\text{Isoc}^\dagger(X, \overline{X})_{\overline{\Sigma}_n(-\text{ss})}$. Let \mathcal{E} be an object in the essential image of (3.9). To see that it is in $\text{Isoc}^\dagger(X, \overline{X})_{\overline{\Sigma}_n(-\text{ss})}$, it suffices to check it locally around generic points of Z . So we may assume that $\overline{X} = \text{Spec } \overline{R}$ is affine, Z is smooth and defined as the zero locus of some element $\bar{t} \in \overline{R}$. Let $M_{\overline{X}}$ be the log structure on \overline{X} associated to Z , let $(\overline{X}_0, M_{\overline{X}_0})$ be the log scheme defined by $\overline{X}_0 := \text{Spec } \overline{R}[\bar{s}]/(\bar{s}^n - \bar{t})$, $M_{\overline{X}_0} := \log$ structure associated to $\{\bar{s} = 0\}$ and let $(\overline{X}_m, M_{\overline{X}_m})$ ($m = 0, 1, 2$) be the $(m + 1)$ -fold fiber product of $(\overline{X}_0, M_{\overline{X}_0})$ over $(\overline{X}, M_{\overline{X}})$. Also, let us put $X_\bullet := X \times_{\overline{X}} \overline{X}_\bullet$. Then the functor (3.9) = (2.54) in this case is equal to the composite

$$\begin{aligned} \text{Isoc}^{\log}((\overline{X}, Z)^{1/n}, M_{(\overline{X}, Z)^{1/n}})_{n\Sigma(-\text{ss})} &\xrightarrow{=} \text{Isoc}^{\log}(\overline{X}_\bullet, M_{\overline{X}_\bullet})_{n\Sigma(-\text{ss})} \\ &\longrightarrow \text{Isoc}^\dagger(X_\bullet, \overline{X}_\bullet) \xleftarrow{=} \text{Isoc}^\dagger(X, \overline{X}). \end{aligned}$$

Hence the restriction of \mathcal{E} to $\text{Isoc}^\dagger(X_0, \overline{X}_0)$ extends to an object in the category $\text{Isoc}^{\log}(\overline{X}_0, M_{\overline{X}_0})_{n\Sigma(-\text{ss})}$. Now let us take a charted smooth standard small frame with genetic point $(\mathbf{X}, t, L) := ((X, \overline{X}, \overline{\mathcal{X}}), t, L)$ enclosing (X, \overline{X}) such that t is a lift of \bar{t} . Then we have a charted smooth standard small frame with genetic point of the form $(\mathbf{X}_0, s, L) := ((X_0, \overline{X}_0, \overline{\mathcal{X}}_0), s, L)$ and a morphism $\psi : (\mathbf{X}_0, s, L) \longrightarrow (\mathbf{X}, t, L)$ with $\psi^*(t) = s^n$ lifting the morphism $\overline{X}_0 \longrightarrow \overline{X}$, by Lemma 1.12 and Remark 1.13. It naturally induces the map $\psi_n : A_L^1[0, 1) \longrightarrow A_L^1[0, 1)$ between ‘discs at generic points’ defined by $t \mapsto t^n$. Let (E, ∇) be the ∇ -module on $A_L^1[\lambda, 1)$ (for some $\lambda \in (0, 1) \cap \Gamma^*$) which is induced by \mathcal{E} . Then, since the restriction of \mathcal{E} to $\text{Isoc}^\dagger(X_0, \overline{X}_0)$ extends to an object (which we denote by $\tilde{\mathcal{E}}$) in $\text{Isoc}^{\log}(\overline{X}_0, M_{\overline{X}_0})_{n\Sigma(-\text{ss})}$, we see that $\psi_n^*(E, \nabla)$ extends to a log- ∇ -module on $A_L^1[0, 1)$ which is induced by $\tilde{\mathcal{E}}$. Then, by Theorem 1.17, $\psi_n^*(E, \nabla)$ is $n\Sigma$ -unipotent ($n\Sigma$ -semisimple). Then $\psi_{n,*}\psi_n^*(E, \nabla)$ is written as a successive extension by the objects of the form (as a direct sum of the objects of the form)

$$\psi_{n,*}(M_{n\xi}, \nabla_{M_{n\xi}}) \cong \bigoplus_{i=0}^{n-1} (M_{\xi + \frac{i}{n}}, \nabla_{M_{\xi + \frac{i}{n}}}) \quad (\xi \in \Sigma).$$

In particular, it is $(\Sigma + \{\frac{i}{n} \mid 0 \leq i \leq n-1\})$ -unipotent ($(\Sigma + \{\frac{i}{n} \mid 0 \leq i \leq n-1\})$ -semisimple). So (E, ∇) , being a direct summand of $\psi_{n,*}\psi_n^*(E, \nabla)$, is also $(\Sigma + \{\frac{i}{n} \mid 0 \leq i \leq n-1\})$ -unipotent ($(\Sigma + \{\frac{i}{n} \mid 0 \leq i \leq n-1\})$ -semisimple). So we have shown that \mathcal{E} has $\overline{\Sigma}_n$ -unipotent ($\overline{\Sigma}_n$ -semisimple) generic monodromy.

Conversely, let \mathcal{E} be an object in $\text{Isoc}^\dagger(X, \overline{X})_{\overline{\Sigma}_n(-\text{ss})}$ and prove that it is in the essential image of (3.9). Since (3.9) is fully faithful, it suffices to prove it Zariski locally. So let $\overline{X}_\bullet, L, \psi_n, \dots$ as in the previous paragraph. Then the ∇ -module (E, ∇) on $A_L^1[\lambda, 1)$ induced by \mathcal{E} is $(\Sigma + \{\frac{i}{n} \mid 0 \leq i \leq n-1\})$ -unipotent ($(\Sigma + \{\frac{i}{n} \mid 0 \leq i \leq n-1\})$ -semisimple). Hence one can see by easy calculation that $\psi_n^*(E, \nabla)$ on $A_L^1[\lambda^{1/n}, 1)$ is $n\Sigma$ -unipotent ($n\Sigma$ -semisimple). So the restriction of \mathcal{E} to $\text{Isoc}^\dagger(X_0, \overline{X}_0)$ has $n\Sigma$ -unipotent ($n\Sigma$ -semisimple) generic monodromy and by Theorem 1.17, it extends uniquely to an object (which we denote by $\tilde{\mathcal{E}}_0$) in $\text{Isoc}^{\log}(\overline{X}_0, M_{\overline{X}_0})_{n\Sigma(-\text{ss})}$. Then, since

a projection $(\overline{X}_l, M_{\overline{X}_l}) \longrightarrow (\overline{X}_0, M_{\overline{X}_0})$ is strict etale (which follows from Proposition 2.20), the pull-back of $\tilde{\mathcal{E}}_0$ to $(\overline{X}_l, M_{\overline{X}_l})$ by this projection is in $\text{Isoc}^{\log}(\overline{X}_l, M_{\overline{X}_l})_{n\Sigma(-ss)}$ by Proposition 1.22. Hence the restriction of \mathcal{E} to $\text{Isoc}^\dagger(X_l, \overline{X}_l)$ has $n\Sigma$ -unipotent ($n\Sigma$ -semisimple) generic monodromy. So it extends uniquely to an object (which we denote by $\tilde{\mathcal{E}}_l$) in $\text{Isoc}^{\log}(\overline{X}_l, M_{\overline{X}_l})_{n\Sigma(-ss)}$. The uniqueness implies that $\tilde{\mathcal{E}}_\bullet$ defines an object in $\text{Isoc}^{\log}(\overline{X}_\bullet, M_{\overline{X}_\bullet})_{n\Sigma(-ss)} = \text{Isoc}^{\log}((\overline{X}, Z)^{1/n}, M_{(\overline{X}, Z)^{1/n}})_{n\Sigma(-ss)}$. Therefore \mathcal{E} is in the essential image of (3.9). So we have shown the equivalence (3.8), hence the equivalence (3.7).

Next we prove that the functor (3.4) induces the equivalence of categories

$$(3.10) \quad \text{Par-Isoc}^{\log}(\overline{X}, Z)_{\Sigma(-ss)} \xrightarrow{=} \text{Isoc}^\dagger(X, \overline{X})_{\overline{\Sigma}(-ss)}.$$

To do so, first we prove the well-definedness and the full faithfulness of the functor (3.10). Let $\tau'_i : (\mathbb{Z}_p/\mathbb{Z}) \setminus ((\Sigma_i + \mathbb{Z}_{(p)})/\mathbb{Z}) \longrightarrow \mathbb{Z}_p$ be any section of the projection $\mathbb{Z}_p \longrightarrow \mathbb{Z}_p/\mathbb{Z}$ and for $a \in \mathbb{Z}_{(p)}$, let $\tau''_{a,i} : (\Sigma_i + \mathbb{Z}_{(p)})/\mathbb{Z} \longrightarrow \mathbb{Z}_p$ be the map defined as $\tau''_{a,i}(\gamma + b) := \gamma + [b]_{-a}$ ($\gamma \in \Sigma_i, b \in \mathbb{Z}_{(p)}$), where $[b]_{-a}$ is the unique element in $(b + \mathbb{Z}) \cap [-a, -a + 1)$. Let $\tau_{a,i} : \mathbb{Z}_p/\mathbb{Z} \longrightarrow \mathbb{Z}_p$ be the map defined as $\tau_{a,i}|_{(\mathbb{Z}_p/\mathbb{Z}) \setminus ((\Sigma_i + \mathbb{Z}_{(p)})/\mathbb{Z})} = \tau'_i, \tau_{a,i}|_{(\Sigma_i + \mathbb{Z}_{(p)})/\mathbb{Z}} = \tau''_{a,i}$ and for $\alpha = (\alpha_i)_i \in \mathbb{Z}_{(p)}^r$, let us put $\tau_\alpha := \prod_{i=1}^r \tau_{\alpha_i, i}$. Let us take $(\mathcal{E}_\alpha)_\alpha \in \text{Par-Isoc}^{\log}(\overline{X}, Z)_{\Sigma(-ss)}$. Then, by (semisimple) Σ -adjustedness of $(\mathcal{E}_\alpha)_\alpha$, \mathcal{E}_α has exponents in $\tau_\alpha(\overline{\Sigma})$ (with semisimple residues). Hence $j^\dagger \mathcal{E}_\alpha$ has $\overline{\Sigma}$ -unipotent ($\overline{\Sigma}$ -semisimple) generic monodromy. So the functor (3.4) naturally induces the functor (3.10), that is, (3.10) is well-defined. Next, let us take $((\mathcal{E}_\alpha)_\alpha, (\iota_{\alpha\beta})_{\alpha,\beta}), ((\mathcal{E}'_\alpha)_\alpha, (\iota'_{\alpha\beta})_{\alpha,\beta}) \in \text{Par-Isoc}^{\log}(\overline{X}, Z)$. Then, since $\mathcal{E}_\alpha, \mathcal{E}'_\alpha$ have exponents in $\tau_\alpha(\overline{\Sigma})$ (with semisimple residues) for any α , we have the isomorphism

$$(3.11) \quad \text{Hom}(\mathcal{E}_\alpha, \mathcal{E}'_\beta) \xrightarrow{=} \text{Hom}(j^\dagger \mathcal{E}_\alpha, j^\dagger \mathcal{E}'_\beta)$$

for any $\alpha = (\alpha_i), \beta = (\beta_i) \in \mathbb{Z}_{(p)}^r$ with $\alpha_i \leq \beta_i$ ($1 \leq i \leq r$), by Proposition 1.18. Using (3.11) in the case $\alpha = \beta$ for $\alpha \in \mathbb{Z}_{(p)}^r$, we see that the functor (3.10) is faithful. On the other hand, if we are given an element $\varphi \in \text{Hom}(j^\dagger \mathcal{E}_\alpha, j^\dagger \mathcal{E}'_\alpha)$ for some α (note that $\text{Hom}(j^\dagger \mathcal{E}_\alpha, j^\dagger \mathcal{E}'_\alpha)$ is independent of α), it induces for any α the unique element $\varphi_\alpha \in \text{Hom}(\mathcal{E}_\alpha, \mathcal{E}'_\alpha)$ which is sent to φ by (3.11) (for $\alpha = \beta$). These φ_α 's satisfy the equality $\iota'_{\alpha\beta} \circ \varphi_\alpha = \varphi_\beta \circ \iota_{\alpha\beta}$ because both are sent to φ by (3.11). Hence $(\varphi_\alpha)_\alpha$ defines a morphism $(\mathcal{E}_\alpha)_\alpha \longrightarrow (\mathcal{E}'_\alpha)_\alpha$ which is sent to φ by the functor (3.10). Hence (3.10) is full, and so we have shown the full faithfulness.

We prove the essential surjectivity of the functor (3.10). Let us take an object \mathcal{E} in $\text{Isoc}^\dagger(X, \overline{X})_{\overline{\Sigma}(-ss)}$ and let τ_α be as in the previous paragraph. Then there exists the unique object $\mathcal{E}_\alpha \in \text{Isoc}^{\log}(\overline{X}, Z)_{\tau_\alpha(\overline{\Sigma})(-ss)}$ with $j^\dagger \mathcal{E}_\alpha = \mathcal{E}$ by Theorem 1.17 and for $\alpha = (\alpha_i)_i, \beta = (\beta_i)_i$ with $\alpha_i \leq \beta_i$, we have the unique morphism $\iota_{\alpha\beta} : \mathcal{E}_\alpha \longrightarrow \mathcal{E}_\beta$ with $j^\dagger \iota_{\alpha\beta} = \text{id}_\mathcal{E}$ by Proposition 1.18. Then $((\mathcal{E}_\alpha)_\alpha, (\iota_{\alpha\beta})_{\alpha,\beta})$ forms an inductive system of objects in $\text{Isoc}^{\log}(\overline{X}, Z)$, and one can check that it satisfies the property (1) in Definition 3.2 by the above-mentioned uniqueness. Also, note that there exists some n prime to p such that \mathcal{E} is actually contained in $\text{Isoc}^\dagger(X, \overline{X})_{\overline{\Sigma}_n(-ss)}$.

Then, for any $\alpha = (\alpha_i)_i$, \mathcal{E}_α is in fact the unique object in $\text{Isoc}^{\log}(\overline{X}, Z)_{\tau_\alpha(\overline{\Sigma}_n)(-\text{ss})}$ with $j^\dagger \mathcal{E}_\alpha = \mathcal{E}$. Since we have $\tau_\alpha(\overline{\Sigma}_n) = \tau_{\alpha'}(\overline{\Sigma}_n)$ when $\alpha' = ([n\alpha_i]/n)_i$, we see that $\iota_{\alpha'\alpha}$ is an isomorphism by uniqueness. So we have checked that $(\mathcal{E}_\alpha)_\alpha$ satisfies the property (2) in Definition 3.2. Also, we see by definition that $\tau_\alpha(\overline{\Sigma}) = \prod_{i=1}^r (\Sigma_i + ([-\alpha_i, -\alpha_i + 1] \cap \mathbb{Z}_{(p)}))$. Hence $(\mathcal{E}_\alpha)_\alpha$ is (semisimply) Σ -adjusted and so we have shown the essential surjectivity of the functor (3.10). Hence the functor (3.10) is an equivalence.

By composing the equivalences (3.7) and (3.10) $^{-1}$, we can construct the equivalence (3.5) which makes the diagram (3.6) commutative. So we are done. \square

Corollary 3.7. *Let $X, \overline{X}, Z = \bigcup_{i=1}^r Z_i$ be as above. Then there exist the canonical equivalences of categories*

$$(3.12) \quad \varinjlim_{(n,p)=1} \text{Isoc}^{\log}((\overline{X}, Z)^{1/n}, M_{(\overline{X}, Z)^{1/n}})_{\mathbf{0}} \xrightarrow{\cong} \text{Par-Isoc}^{\log}(\overline{X}, Z)_{\mathbf{0}},$$

$$(3.13) \quad \varinjlim_{(n,p)=1} \text{Isoc}((\overline{X}, Z)^{1/n}) \xrightarrow{\cong} \text{Par-Isoc}^{\log}(\overline{X}, Z)_{\mathbf{0-ss}},$$

$$(3.14) \quad \varinjlim_{(n,p)=1} F\text{-Isoc}^{\log}((\overline{X}, Z)^{1/n}, M_{(\overline{X}, Z)^{1/n}}) \xrightarrow{\cong} \text{Par-}F\text{-Isoc}^{\log}(\overline{X}, Z)_{\mathbf{0}},$$

$$(3.15) \quad \varinjlim_{(n,p)=1} F\text{-Isoc}((\overline{X}, Z)^{1/n}) \xrightarrow{\cong} \text{Par-}F\text{-Isoc}^{\log}(\overline{X}, Z)_{\mathbf{0-ss}},$$

$$(3.16) \quad \varinjlim_{(n,p)=1} F\text{-Isoc}((\overline{X}, Z)^{1/n})^\circ \xrightarrow{\cong} \text{Par-}F\text{-Isoc}^{\log}(\overline{X}, Z)_{\mathbf{0-ss}}^\circ.$$

In particular, we have the canonical equivalence

$$(3.17) \quad \text{Rep}_{K^\sigma}(\pi_1(X)) \xrightarrow{\cong} \varinjlim_{(n,p)=1} F\text{-Isoc}((\overline{X}, Z)^{1/n})^\circ \xrightarrow{\cong} \text{Par-}F\text{-Isoc}^{\log}(\overline{X}, Z)_{\mathbf{0-ss}}^\circ$$

when X is connected.

The equivalence (3.17) is nothing but (0.10), which is a p -adic version of (0.3).

Proof. The equivalences (3.12) and (3.13) are special cases of Theorem 3.6 (note that we have $\text{Isoc}^{\log}((\overline{X}, Z)^{1/n}, M_{(\overline{X}, Z)^{1/n}})_{\mathbf{0-ss}} = \text{Isoc}((\overline{X}, Z)^{1/n})$). Next, let $F\text{-Isoc}^\dagger(X, \overline{X})_{(\mathbb{Z}_{(p)}/\mathbb{Z})^r(-\text{ss})}$ be the category of pairs (\mathcal{E}, Ψ) consisting of $\mathcal{E} \in \text{Isoc}^\dagger(X, \overline{X})_{(\mathbb{Z}_{(p)}/\mathbb{Z})^r(-\text{ss})}$ endowed with the isomorphism $\Psi : F^* \mathcal{E} \xrightarrow{\cong} \mathcal{E}$ and let $F\text{-Isoc}^\dagger(X, \overline{X})_{(\mathbb{Z}_{(p)}/\mathbb{Z})^r(-\text{ss})}^\circ$ be the category of pairs $(\mathcal{E}, \Psi) \in F\text{-Isoc}^\dagger(X, \overline{X})_{(\mathbb{Z}_{(p)}/\mathbb{Z})^r(-\text{ss})}$ which is unit-root as an object in

$F\text{-Isoc}^\dagger(X, \overline{X})$. Then it suffices to prove the equivalences

$$(3.18) \quad \varinjlim_{(n,p)=1} F\text{-Isoc}^{\log}((\overline{X}, Z)^{1/n}, M_{(\overline{X}, Z)^{1/n}}) \xrightarrow{=} F\text{-Isoc}^\dagger(X, \overline{X})_{(\mathbb{Z}_{(p)}/\mathbb{Z})^r},$$

$$(3.19) \quad \varinjlim_{(n,p)=1} F\text{-Isoc}((\overline{X}, Z)^{1/n}) \xrightarrow{=} F\text{-Isoc}^\dagger(X, \overline{X})_{(\mathbb{Z}_{(p)}/\mathbb{Z})^{r\text{-ss}}},$$

$$(3.20) \quad \varinjlim_{(n,p)=1} F\text{-Isoc}((\overline{X}, Z)^{1/n})^\circ \xrightarrow{=} F\text{-Isoc}^\dagger(X, \overline{X})_{(\mathbb{Z}_{(p)}/\mathbb{Z})^{r\text{-ss}}}^\circ,$$

$$(3.21) \quad \text{Par-}F\text{-Isoc}^{\log}(\overline{X}, Z)_\mathbf{0} \xrightarrow{=} F\text{-Isoc}^\dagger(X, \overline{X})_{(\mathbb{Z}_{(p)}/\mathbb{Z})^r},$$

$$(3.22) \quad \text{Par-}F\text{-Isoc}^{\log}(\overline{X}, Z)_{\mathbf{0}\text{-ss}} \xrightarrow{=} F\text{-Isoc}^\dagger(X, \overline{X})_{(\mathbb{Z}_{(p)}/\mathbb{Z})^{r\text{-ss}}},$$

$$(3.23) \quad \text{Par-}F\text{-Isoc}^{\log}(\overline{X}, Z)^\circ \xrightarrow{=} F\text{-Isoc}^\dagger(X, \overline{X})_{(\mathbb{Z}_{(p)}/\mathbb{Z})^{r\text{-ss}}}^\circ.$$

As a direct consequence of the equivalence (3.7) (and the functoriality of it with respect to Frobenius), we obtain the equivalence

$$\varinjlim_{(n,p)=1} F\text{-Isoc}^{\log}((\overline{X}, Z)^{1/n}, M_{(\overline{X}, Z)^{1/n}})_\mathbf{0} \xrightarrow{=} F\text{-Isoc}^\dagger(X, \overline{X})_{(\mathbb{Z}_{(p)}/\mathbb{Z})^r}$$

and the equivalence (3.19) (note that $F\text{-Isoc}^{\log}((\overline{X}, Z)^{1/n}, M_{(\overline{X}, Z)^{1/n}})_\mathbf{0} = F\text{-Isoc}((\overline{X}, Z)^{1/n})$.) To prove the equivalence (3.18), it suffice to prove the equivalence

$$(3.24) \quad F\text{-Isoc}^{\log}((\overline{X}, Z)^{1/n}, M_{(\overline{X}, Z)^{1/n}})_\mathbf{0} \xrightarrow{=} F\text{-Isoc}^{\log}((\overline{X}, Z)^{1/n}, M_{(\overline{X}, Z)^{1/n}}),$$

that is, any object (\mathcal{E}, Ψ) in $F\text{-Isoc}^{\log}((\overline{X}, Z)^{1/n}, M_{(\overline{X}, Z)^{1/n}})$ has exponents in $\mathbf{0}$. By definition of exponents given in Definition 2.9, we can check it by pulling (\mathcal{E}, Ψ) back to some fine log scheme (Y, M_Y) such that Y is smooth over k and that M_Y is associated to some simple normal crossing divisor Z in Y . Moreover, by Lemma 1.15(2), we may assume that Z is a non-empty smooth divisor and we may shrink Y if necessary (as long as Z is non-empty). So we may assume that there exists a charted smooth standard small frame $((Y \setminus Z, Y, \mathcal{Y}), t)$ enclosing $(Y, Y \setminus Z)$, and that, if we put $\mathcal{Z} := \{t = 0\}$, there exists a σ^* -linear endomorphism $\varphi : (\mathcal{Y}, \mathcal{Z}) \rightarrow (\mathcal{Y}, \mathcal{Z})$ as fine log formal scheme lifting the q -th power map $F : (Y, Z) \rightarrow (Y, Z)$ with $\varphi^*t = t^q$. Then \mathcal{E} gives rise to a log- ∇ -module (E, ∇) on \mathcal{Y}_K with respect to t and the isomorphism $\Psi : F^*\mathcal{E} \xrightarrow{=} \mathcal{E}$ gives rise to an isomorphism

$$(3.25) \quad \Psi : \varphi^*(E, \nabla) \xrightarrow{=} (E, \nabla).$$

Let us denote the residue of (E, ∇) along $\mathcal{Z}_K = \{t = 0\}$ by res and let $P(x) \in K[x]$ be the minimal polynomial of res . Then it suffices to prove that all the roots of $P(x)$ are 0. Let $a \geq 0$ be the maximum of the absolute values of the roots of $P(x)$. By the isomorphism (3.25), we have the equality $p\Psi(\varphi^*(\text{res})) = \text{res}$, and so we have $P^\sigma(\text{res}/p) = 0$. Hence $P(x)$ divides $P^\sigma(x/p)$, and so we have $a \leq q^{-1}a$, that is, $a = 0$. Hence all the roots of $P(x)$ are 0 as desired and so we have shown the equivalence (3.24), thus the equivalence (3.18).

To prove the equivalence (3.20), it suffices to prove that an object (\mathcal{E}, Ψ) in $F\text{-Isoc}((\overline{X}, Z)^{1/n})$ which is unit-root in $F\text{-Isoc}^\dagger(X, \overline{X})$ is necessarily unit-root. To prove this, we may work locally. So we can take a surjective etale morphism $\overline{X}_0 \rightarrow (\overline{X}, Z)^{1/n}$ with $\overline{X}_0 \in \text{Sch}$. To prove the unit-rootness on (\mathcal{E}, Ψ) , we need to prove that, for any $Y \rightarrow (\overline{X}, Z)^{1/n}$ with $Y \in \text{Sch}$ and for any perfect-field-valued point $y \rightarrow Y$ of Y , the Newton polygon of the F -isocrystal (\mathcal{E}_y, Ψ_y) on $K_y = K \otimes_{W(k)} W(k(y))$ induced by (\mathcal{E}, Ψ) has pure slope 0. Since we may check it etale locally on y , we may replace Y by $\overline{X}_0 \times_X Y$. So it suffices to check it for perfect-field-valued points $y \rightarrow \overline{X}_0'$ of \overline{X}_0 . Moreover, it suffices to check it for perfect-field-valued points $y \rightarrow \overline{X}_0'$ of \overline{X}_0' which admits a surjective morphism of finite type $\overline{X}_0' \rightarrow \overline{X}_0$. So it suffices to prove that the restriction of (\mathcal{E}, Ψ) to $F\text{-Isoc}(\overline{X}_0')$ for such \overline{X}_0' is unit-root. Since the restriction of (\mathcal{E}, Ψ) to $F\text{-Isoc}^\dagger(X, \overline{X})$ is unit-root, so is the restriction of (\mathcal{E}, Ψ) to $F\text{-Isoc}^\dagger(X_0, \overline{X}_0)$. Then, by [49], there exists an alteration $\overline{X}_0' \rightarrow \overline{X}_0$ such that the restriction of (\mathcal{E}, Ψ) to $F\text{-Isoc}^\dagger(X_0', \overline{X}_0')$ (where $X_0' := X \times_{\overline{X}_0} \overline{X}_0'$) extends to a unit-root object $(\tilde{\mathcal{E}}, \tilde{\Psi})$ in $F\text{-Isoc}(\overline{X}_0')$. By the full faithfulness of $F\text{-Isoc}(\overline{X}_0') \rightarrow F\text{-Isoc}^\dagger(X_0', \overline{X}_0')$, $(\tilde{\mathcal{E}}, \tilde{\Psi})$ coincides with the restriction of (\mathcal{E}, Ψ) to $F\text{-Isoc}(\overline{X}_0')$. So the restriction of (\mathcal{E}, Ψ) to $F\text{-Isoc}(\overline{X}_0')$ is unit-root and so we have shown the equivalence (3.20).

Let us prove the equivalences (3.21), (3.22). The faithfulness of them follows from the equivalence (3.10). To prove the fullness, let us take $((\mathcal{E}_\alpha)_\alpha, \Psi), ((\mathcal{E}'_\alpha)_\alpha, \Psi') \in \text{Par-}F\text{-Isoc}^{\log}(\overline{X}, Z)_{\mathbf{0}(-\text{ss})}$ and assume we are given a morphism $f : (j^\dagger \mathcal{E}_\alpha, j^\dagger \Psi) \rightarrow (j^\dagger \mathcal{E}'_\alpha, j^\dagger \Psi')$. Then, by the equivalence (3.10), we have the unique morphism

$$\tilde{f} = (\tilde{f}_\alpha)_\alpha : (\mathcal{E}_\alpha)_\alpha \rightarrow (\mathcal{E}'_\alpha)_\alpha$$

lifting f . Let us see that \tilde{f} is compatible with Ψ and Ψ' . Take any $\alpha = (\alpha_i) \in \mathbb{Z}_{(p)}^r$ and take $\beta = (\beta_i)$ with $q\alpha_i \leq \beta_i$ such that Ψ, Ψ' induce morphisms $\Psi_\alpha : F^* \mathcal{E}_\alpha \rightarrow \mathcal{E}_\beta, \Psi'_\alpha : F^* \mathcal{E}'_\alpha \rightarrow \mathcal{E}'_\beta$. Under this situation, it suffices to prove the equality $\tilde{f}_\beta \circ \Psi_\alpha = \Psi'_\alpha \circ (F^* \tilde{f}_\alpha)$. This follows from the equality

$$j^\dagger(\tilde{f}_\beta \circ \Psi_\alpha) = f_\beta \circ j^\dagger \Psi_\alpha \stackrel{\text{assumption}}{=} j^\dagger \Psi'_\alpha \circ (F^* f_\alpha) = j^\dagger(\Psi'_\alpha \circ (F^* \tilde{f}_\alpha))$$

and Proposition 1.18, since the exponents of $F^* \mathcal{E}_\alpha$ is contained in $q\tau_\alpha((\mathbb{Z}_{(p)}/\mathbb{Z})^r)$, where τ_α is as in the proof of Theorem 3.6. So we have shown that (3.21), (3.22) are fully faithful. To prove the essential surjectivity, let us take an object (\mathcal{E}, Ψ) in $F\text{-Isoc}^\dagger(X, \overline{X})_{(\mathbb{Z}_{(p)}/\mathbb{Z})^r(-\text{ss})}$ and define $(\mathcal{E}_\alpha)_\alpha \in \text{Par-Isoc}^{\log}(\overline{X}, Z)_{\mathbf{0}(-\text{ss})}$ which is sent to \mathcal{E} as in the proof of Theorem 3.6. Then, for any $\alpha \in \mathbb{Z}_{(p)}^r$, we have the unique morphism $\Psi_\alpha : F^* \mathcal{E}_\alpha \rightarrow \mathcal{E}_{q\alpha}$ extending Ψ , by Proposition 1.18. On the other hand, for any $\alpha = (\alpha_i)_i \in \mathbb{Z}_{(p)}^r$, we have the unique morphism $\Psi_\alpha^{-1} : \mathcal{E}_\alpha \rightarrow F^* \mathcal{E}_\beta$ extending Ψ^{-1} when $\beta = (\beta_i)_i \in \mathbb{Z}_{(p)}^r$ satisfies $\beta_i \geq (\alpha_i/q) + 1$, again by Proposition 1.18. Again by using Proposition 1.18, we see that $\tilde{\Psi} = (\Psi_\alpha)_\alpha, \tilde{\Psi}^{-1} = (\Psi_\alpha^{-1})_\alpha$ define morphisms as ind-objects which are the inverse of each other. So $((\mathcal{E}_\alpha)_\alpha, \tilde{\Psi})$ defines an object

in $\text{Par-}F\text{-Isoc}^{\log}(\overline{X}, Z)_{\mathbf{0}(-\text{ss})}$ which is sent to (\mathcal{E}, Ψ) . So we have shown that (3.21), (3.22) are essentially surjective and so they are equivalences. The equivalence (3.23) is the immediate consequence of the equivalence (3.22). \square

Finally in this section, we prove a proposition which we need to finish the proof of Propositions 2.28, 2.29.

Proposition 3.8. *Let \overline{X} be a connected proper smooth curve of genus 0 over k , let $Z \subseteq \overline{X}$ be a k -rational point and let $X = \overline{X} \setminus Z$. Let Σ be a subset of \mathbb{Z}_p which is (NRD) and (SNLD). Then, if $\Sigma \cap \mathbb{Z}_{(p)}$ is empty, the category $\varinjlim_{(n,p)=1} \text{Isoc}^{\log}((\overline{X}, Z)^{1/n}, M_{(\overline{X}, Z)^{1/n}})_{n\Sigma}$ is empty. If $\Sigma \cap \mathbb{Z}_{(p)}$ consists of one element, the functor*

$$\varinjlim_{(n,p)=1} \text{Isoc}^{\log}((\overline{X}, Z)^{1/n}, M_{(\overline{X}, Z)^{1/n}})_{n\Sigma} \xrightarrow{(3.7)} \text{Isoc}^{\dagger}(X, \overline{X})_{\overline{\Sigma}} \longrightarrow \text{Vect}_K$$

(where $\overline{\Sigma} := \text{Im}(\Sigma + \mathbb{Z}_{(p)} \hookrightarrow \mathbb{Z}_p \longrightarrow \mathbb{Z}_p/\mathbb{Z})$ and the last functor is defined by $\mathcal{E} \mapsto \text{Hom}(j^{\dagger}\mathcal{O}, \mathcal{E})$ with $j^{\dagger}\mathcal{O}$ the structure overconvergent isocrystal on (X, \overline{X}) over K) is an equivalence.

Proof. By Theorem 3.6, it suffices to prove the same property for the category $\text{Par-Isoc}^{\log}(\overline{X}, Z)_{\Sigma}$ (the functor (3.7) replaced by (3.10)). Let $\mathbf{X}_K \hookrightarrow \overline{\mathbf{X}}_K \leftarrow \mathbf{Z}_K, \mathcal{X} \hookrightarrow \overline{\mathcal{X}} \leftarrow \mathcal{Z}, \text{LNM}_{(\overline{\mathbf{X}}_K, \mathbf{Z}_K), ?} (? \subseteq \mathbb{Z}_p)$ be as in the proof of Proposition 2.29. Let us take an object $\mathcal{E} = (\mathcal{E}_{\alpha})_{\alpha} \in \text{Par-Isoc}^{\log}(\overline{X}, Z)_{\Sigma}$. Then \mathcal{E}_0 , which is an object in $\text{Isoc}^{\log}(\overline{X}, Z)_{\Sigma + ([0,1] \cap \mathbb{Z}_{(p)})}$, induces naturally an object (E, ∇) in $\text{LNM}_{(\overline{\mathcal{X}}, \mathcal{Z}), \Sigma + ([0,1] \cap \mathbb{Z}_{(p)})} \cong \text{LNM}_{(\overline{\mathbf{X}}_K, \mathbf{Z}_K), \Sigma + ([0,1] \cap \mathbb{Z}_{(p)})}$. Then, by claim in the proof of Proposition 2.29, there is no such object if $(\Sigma + ([0,1] \cap \mathbb{Z}_{(p)})) \cap \mathbb{Z}$ is empty, that is, if $\Sigma \cap \mathbb{Z}_{(p)}$ is empty. So we can conclude that $\text{Par-Isoc}^{\log}(\overline{X}, Z)_{\Sigma}$ is empty and we are done in this case. If $\Sigma \cap \mathbb{Z}_{(p)}$ consists of one element, $(\Sigma + ([0,1] \cap \mathbb{Z}_{(p)})) \cap \mathbb{Z} = \{N\}$ also consists of one element. Then, by claim in the proof of Proposition 2.29, (E, ∇) is a finite direct sum of $(\mathcal{O}_{\overline{\mathbf{X}}_K}(-N\mathbf{Z}_K), d)$. So the restriction of (E, ∇) (regarded as an object in $\text{LNM}_{(\overline{\mathcal{X}}, \mathcal{Z}), \Sigma + ([0,1] \cap \mathbb{Z}_{(p)})}$) to a strict neighborhood of \mathcal{X}_K in $\overline{\mathcal{X}}_K$ is a finite direct sum of the structure overconvergent isocrystal on (X, \overline{X}) . Hence the object \mathcal{E} is sent by (3.10) to a finite direct sum of the structure overconvergent isocrystal on (X, \overline{X}) . So the functor

$$\text{Par-Isoc}^{\log}(\overline{X}, Z)_{\Sigma} \xrightarrow{(3.10)} \text{Isoc}^{\dagger}(X, \overline{X})_{\overline{\Sigma}} \longrightarrow \text{Vect}_K$$

(the last functor is defined by $\mathcal{E} \mapsto \text{Hom}(j^{\dagger}\mathcal{O}, \mathcal{E})$) is an equivalence in this case. So the proof is finished. \square

4 Unit-root F -lattices

In this section, we will prove the equivalences (0.11), (0.12) and (0.13). To do so, first we recall the definition of unit-root F -lattices and recall how the equivalence of Crew (0.6) is factorized in liftable case.

Let \mathcal{X}_\circ be a p -adic formal scheme smooth and separated of finite type over $\mathrm{Spf} W(k)$ endowed with a lift $F_\circ : \mathcal{X}_\circ \rightarrow \mathcal{X}_\circ$ of the q -th power Frobenius endomorphism on the special fiber $\mathcal{X}_\circ \otimes_{W(k)} k$ which is compatible with $(\sigma|_{W(k)})^* : \mathrm{Spf} W(k) \rightarrow \mathrm{Spf} W(k)$, and let us put $\mathcal{X} := \mathcal{X}_\circ \otimes_{W(k)} O_K$, $F := F_\circ \otimes_{(\sigma|_{W(k)})^*} \sigma^* : \mathcal{X} \rightarrow \mathcal{X}$. Then an F -lattice (resp. a unit-root F -lattice) on \mathcal{X} is defined to be a pair (\mathcal{E}, ϕ) consisting of a locally free $\mathcal{O}_\mathcal{X}$ -module \mathcal{E} of finite rank and an isomorphism $\phi : (F^*\mathcal{E})_\mathbb{Q} \xrightarrow{\cong} \mathcal{E}_\mathbb{Q}$ in the \mathbb{Q} -linearization $\mathrm{Coh}(\mathcal{X})_\mathbb{Q}$ of the category $\mathrm{Coh}(\mathcal{X})$ of coherent $\mathcal{O}_\mathcal{X}$ -modules. (resp. an isomorphism $\phi : F^*\mathcal{E} \xrightarrow{\cong} \mathcal{E}$ in the category $\mathrm{Coh}(\mathcal{X})$ of coherent $\mathcal{O}_\mathcal{X}$ -modules.) (An unit-root F -lattice on \mathcal{X} is called a unit-root F -lattice on $\mathcal{X}_\circ/(O_K, \sigma)$ in [9].) Let us denote the category of F -lattices (resp. unit-root F -lattices) on \mathcal{X} by $F\text{-Latt}(\mathcal{X})$ (resp. $F\text{-Latt}(\mathcal{X})^\circ$). (Note that the same definition is possible even when \mathcal{X}_\circ is a diagram of p -adic formal schemes smooth and separated of finite type over $\mathrm{Spf} W(k)$ endowed with an endomorphism $F_\circ : \mathcal{X}_\circ \rightarrow \mathcal{X}_\circ$ as above. Note also that only the unit-root F -lattices are treated in this section and the F -lattices will be treated in the next section.)

Let $\mathcal{X}_\circ, \mathcal{X}, F$ be as above and let us put $X := \mathcal{X}_\circ \otimes_{W(k)} k$. Then, the equivalence (0.6) of Crew is written as the composite of the equivalence of Katz ([17, 4.1.1], [9, 2.2])

$$(4.1) \quad \mathrm{Rep}_{O_K^\sigma}(\pi_1(X)) \xrightarrow{\cong} F\text{-Latt}(\mathcal{X})^\circ$$

and the equivalence

$$(4.2) \quad F\text{-Latt}(\mathcal{X})_\mathbb{Q}^\circ \xrightarrow{\cong} F\text{-Isoc}(X)^\circ$$

proved in [9], where, for an additive category \mathcal{C} , $\mathcal{C}_\mathbb{Q}$ denotes the \mathbb{Q} -linearization of it.

Let us recall the definition of (4.1) (cf. Section 2.2). Let ρ be an object in $\mathrm{Rep}_{O_K^\sigma}(\pi_1(X))$ and let \mathcal{F} be the corresponding object in $\mathrm{Sm}_{O_K^\sigma}(X)$. Let $\tilde{\mathcal{F}}$ be the object in $\mathrm{Sm}_{O_K^\sigma}(\mathcal{X})$ corresponding to \mathcal{F} via the equivalence $\mathrm{Sm}_{O_K^\sigma}(\mathcal{X}) \cong \mathrm{Sm}_{O_K^\sigma}(X)$. Then $F : \mathcal{X} \rightarrow \mathcal{X}$ induces the equivalence $F^{-1} : \mathrm{Sm}_{O_K^\sigma}(\mathcal{X}) \xrightarrow{\cong} \mathrm{Sm}_{O_K^\sigma}(\mathcal{X})$ with $F^{-1}(\tilde{\mathcal{F}}) \cong \tilde{\mathcal{F}}$. Then, if we define \mathcal{E} and ϕ by $\mathcal{E} := \tilde{\mathcal{F}} \otimes_{O_K^\sigma} \mathcal{O}_\mathcal{X}$,

$$\phi : F^*\mathcal{E} = F^{-1}\tilde{\mathcal{F}} \otimes_{O_K^\sigma} \mathcal{O}_\mathcal{X} \xrightarrow{\cong} \tilde{\mathcal{F}} \otimes_{O_K^\sigma} \mathcal{O}_\mathcal{X} = \mathcal{E},$$

the functor (4.1) is defined as $\rho \mapsto (\mathcal{E}, \phi)$. In view of this, we see that (4.1) is written as the equivalence

$$(4.3) \quad \mathrm{Sm}_{O_K^\sigma}(X) \xrightarrow{\cong} F\text{-Latt}(\mathcal{X})^\circ.$$

(The choice of a base point in the definition of $\pi_1(X)$ is not essential.) Also, we see easily the functoriality of (4.1), (4.3).

Now we fix the notation to prove the equivalences (0.11), (0.12) and (0.13). In the following in this section, let $X \hookrightarrow \overline{X}$ be an open immersion of connected smooth

k -varieties such that $\overline{X} \setminus X =: Z = \bigcup_{i=1}^r Z_i$ is a simple normal crossing divisor (with each Z_i irreducible), and assume that we have an open immersion $\mathcal{X}_\circ \hookrightarrow \overline{\mathcal{X}}_\circ$ of p -adic formal schemes smooth and separated of finite type over $\mathrm{Spf} W(k)$ such that there exists a relative simple normal crossing divisor $\mathcal{Z}_\circ := \bigcup_{i=1}^r \mathcal{Z}_{\circ,i} \hookrightarrow \overline{\mathcal{X}}_\circ$ with $\overline{\mathcal{X}}_\circ \setminus \mathcal{Z}_\circ = \mathcal{X}_\circ$, $\overline{\mathcal{X}}_\circ \otimes_{W(k)} k = \overline{X}$, $\mathcal{Z}_{\circ,i} \otimes_{W(k)} k = Z_i$. Assume moreover that we have a lift $F_\circ : (\overline{\mathcal{X}}_\circ, \mathcal{Z}_\circ) \rightarrow (\overline{\mathcal{X}}_\circ, \mathcal{Z}_\circ)$ (endomorphism as log schemes) of the q -th power Frobenius endomorphism on (\overline{X}, Z) which is compatible with $(\sigma|_{W(k)})^* : \mathrm{Spf} W(k) \rightarrow \mathrm{Spf} W(k)$. Let us put $\mathcal{X} := \mathcal{X}_\circ \otimes_{W(k)} O_K$, $\overline{\mathcal{X}} := \overline{\mathcal{X}}_\circ \otimes_{W(k)} O_K$, $\mathcal{Z} := \mathcal{Z}_\circ \otimes_{W(k)} O_K$, $\mathcal{Z}_i := \mathcal{Z}_{\circ,i} \otimes_{W(k)} O_K$. Then $(\overline{\mathcal{X}}, \mathcal{Z})$ admits the lift $F := F_\circ \otimes_{(\sigma|_{W(k)})^*} \sigma^* : (\overline{\mathcal{X}}, \mathcal{Z}) \rightarrow (\overline{\mathcal{X}}, \mathcal{Z})$ of the q -th power Frobenius endomorphism on (\overline{X}, Z) which is compatible with $\sigma^* : \mathrm{Spf} O_K \rightarrow \mathrm{Spf} O_K$.

For $a \in \mathbb{N}$, let us put $(\overline{\mathcal{X}}_\circ, \mathcal{Z}_\circ)_a := (\overline{\mathcal{X}}_{\circ,a}, \mathcal{Z}_{\circ,a}) := (\overline{\mathcal{X}}, \mathcal{Z}) \otimes_{W(k)} W_a(k)$, $\mathcal{Z}_{\circ,i,a} := \mathcal{Z}_{\circ,i} \otimes_{W(k)} W_a(k)$, $(\overline{\mathcal{X}}, \mathcal{Z})_a := (\overline{\mathcal{X}}_a, \mathcal{Z}_a) := (\overline{\mathcal{X}}, \mathcal{Z}) \otimes_{W(k)} W_a(k)$. As in Section 2, let \mathcal{G}_X^t be the category of finite etale Galois tame covering of X (tamely ramified at generic points of Z) and for an object $Y \rightarrow X$ in \mathcal{G}_X^t , let \overline{Y} be the normalization of \overline{X} in $k(Y)$, let $\overline{Y}^{\mathrm{sm}}$ be the smooth locus of \overline{Y} and let $G_Y := \mathrm{Aut}(Y/X)$. Then \overline{Y} admits naturally the log structure $M_{\overline{Y}}$ associated to $\overline{Y} \setminus Y$ such that the morphism $(\overline{Y}, M_{\overline{Y}}) \rightarrow (\overline{X}, \mathcal{Z})$ is finite Kummer log etale ([13]). Then this morphism admits a unique finite Kummer log etale lifts $(\overline{\mathcal{Y}}_{\circ,a}, M_{\overline{\mathcal{Y}}_{\circ,a}}) \rightarrow (\overline{\mathcal{X}}_\circ, \mathcal{Z}_\circ)_a$ for $a \in \mathbb{N}$ (hence the lift $(\overline{\mathcal{Y}}_\circ, M_{\overline{\mathcal{Y}}_\circ}) := \varinjlim_a (\overline{\mathcal{Y}}_{\circ,a}, M_{\overline{\mathcal{Y}}_{\circ,a}}) \rightarrow (\overline{\mathcal{X}}_\circ, \mathcal{Z}_\circ)$) by [13, 2.8, 3.10, 3.11]. Let us put $(\overline{\mathcal{Y}}, M_{\overline{\mathcal{Y}}}) := (\overline{\mathcal{Y}}_\circ, M_{\overline{\mathcal{Y}}_\circ}) \otimes_{W(k)} O_K$ and let $\overline{\mathcal{Y}}_\circ^{\mathrm{sm}} \subseteq \overline{\mathcal{Y}}_\circ$, $\overline{\mathcal{Y}}^{\mathrm{sm}} \subseteq \overline{\mathcal{Y}}$ be the smooth loci. Then the endomorphism $F_\circ : (\overline{\mathcal{X}}_\circ, \mathcal{Z}_\circ) \rightarrow (\overline{\mathcal{X}}_\circ, \mathcal{Z}_\circ)$ uniquely lifts to an endomorphism F_\circ on $(\overline{\mathcal{Y}}_\circ, M_{\overline{\mathcal{Y}}_\circ})$ and on $(\overline{\mathcal{Y}}_\circ^{\mathrm{sm}}, M_{\overline{\mathcal{Y}}_\circ^{\mathrm{sm}}} := M_{\overline{\mathcal{Y}}_\circ}|_{\overline{\mathcal{Y}}_\circ^{\mathrm{sm}}})$, and it induces the endomorphism $F := F_\circ \otimes \sigma^*$ on $(\overline{\mathcal{Y}}, M_{\overline{\mathcal{Y}}})$ and on $(\overline{\mathcal{Y}}^{\mathrm{sm}}, M_{\overline{\mathcal{Y}}^{\mathrm{sm}}} := M_{\overline{\mathcal{Y}}}|_{\overline{\mathcal{Y}}^{\mathrm{sm}}})$. So the category $F\text{-Latt}(\overline{\mathcal{Y}}^{\mathrm{sm}})^\circ$ of unit-root F -lattices on $\overline{\mathcal{Y}}^{\mathrm{sm}}$ is defined.

Let us put $(\overline{\mathcal{Y}}_a, M_{\overline{\mathcal{Y}}_a}) := (\overline{\mathcal{Y}}, M_{\overline{\mathcal{Y}}}) \otimes_{W(k)} W_a(k)$, $(\overline{\mathcal{Y}}_a^{\mathrm{sm}}, M_{\overline{\mathcal{Y}}_a^{\mathrm{sm}}}) := (\overline{\mathcal{Y}}^{\mathrm{sm}}, M_{\overline{\mathcal{Y}}^{\mathrm{sm}}}) \otimes_{W(k)} W_a(k)$. Then G_Y naturally acts on them and so we can define the fine log Deligne-Mumford stack $([\overline{\mathcal{Y}}_a/G_Y], M_{[\overline{\mathcal{Y}}_a/G_Y]})$ as in Example 2.14. As an open substack of it, we have the fine log Deligne-Mumford stack $([\overline{\mathcal{Y}}_a^{\mathrm{sm}}/G_Y], M_{[\overline{\mathcal{Y}}_a^{\mathrm{sm}}/G_Y]})$ for $a \in \mathbb{N}$ and they induce the ind fine log algebraic stack $([\overline{\mathcal{Y}}^{\mathrm{sm}}/G_Y], M_{[\overline{\mathcal{Y}}^{\mathrm{sm}}/G_Y]}) := \varinjlim_a ([\overline{\mathcal{Y}}_a^{\mathrm{sm}}/G_Y], M_{[\overline{\mathcal{Y}}_a^{\mathrm{sm}}/G_Y]})$. Note also that the endomorphism F on $(\overline{\mathcal{Y}}^{\mathrm{sm}}, M_{\overline{\mathcal{Y}}^{\mathrm{sm}}})$ induces the endomorphism on $([\overline{\mathcal{Y}}^{\mathrm{sm}}/G_Y], M_{[\overline{\mathcal{Y}}^{\mathrm{sm}}/G_Y]})$, which we denote also by F . Then we define the category $F\text{-Latt}([\overline{\mathcal{Y}}^{\mathrm{sm}}/G_Y])^\circ$ of unit-root F -lattices on $[\overline{\mathcal{Y}}^{\mathrm{sm}}/G_Y]$ as the category of pairs (\mathcal{E}, ϕ) consisting of a locally free module \mathcal{E} of finite rank on $[\overline{\mathcal{Y}}^{\mathrm{sm}}/G_Y]$ (= a compatible family of locally free modules of finite rank on $[\overline{\mathcal{Y}}_a^{\mathrm{sm}}/G_Y]$ for $a \in \mathbb{N}$) and an isomorphism $\phi : F^* \mathcal{E} \xrightarrow{\sim} \mathcal{E}$. It is equivalent to the category of objects in $F\text{-Latt}(\overline{\mathcal{Y}}^{\mathrm{sm}})^\circ$ endowed with equivariant action of G_Y . By Katz' equivalence (4.1) for $\overline{\mathcal{Y}}^{\mathrm{sm}}$, we have the equivalence

$$(4.4) \quad \mathrm{Rep}_{O_K^\sigma}(\pi_1(\overline{\mathcal{Y}}^{\mathrm{sm}})) \xrightarrow{\sim} F\text{-Latt}(\overline{\mathcal{Y}}^{\mathrm{sm}})^\circ,$$

and the left hand side is unchanged if we replace $\overline{\mathcal{Y}}^{\mathrm{sm}}$ by its open formal sub-

scheme $\overline{Y}^{\text{sm}'}$ with $\text{codim}(\overline{Y}^{\text{sm}} \setminus \overline{Y}^{\text{sm}'}, \overline{Y}^{\text{sm}}) \geq 2$. Hence the right hand side is also unchanged if we replace \overline{Y}^{sm} by its open formal subscheme $\overline{Y}^{\text{sm}'}$ with $\text{codim}(\overline{Y}^{\text{sm}} \setminus \overline{Y}^{\text{sm}'}, \overline{Y}^{\text{sm}}) \geq 2$. Using this, we see by the argument in the proof of Proposition 2.16 that the limit $\varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} F\text{-Latt}([\overline{Y}^{\text{sm}}/G_Y])^\circ$ is defined. Also, let $\text{Vect}([\overline{Y}^{\text{sm}}/G_Y])$ be the category of locally free modules of finite rank on $[\overline{Y}^{\text{sm}}/G_Y]$. Then the limit $\varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} \text{Vect}([\overline{Y}^{\text{sm}}/G_Y])$ is defined when X is a curve.

On the other hand, we have the morphism $\overline{\mathcal{X}}_{\circ,a} \rightarrow [\mathbb{A}_{W_a(k)}^r/\mathbb{G}_{m,W_a(k)}^r]$ defined by the log structure $M_{\overline{\mathcal{X}}_a}$ on $\overline{\mathcal{X}}_{\circ,a}$ associated to $\mathcal{Z}_{\circ,a}$ and the natural homomorphism $\mathbb{N}^r \rightarrow \overline{M}_{\overline{\mathcal{X}}_{\circ,a}} \cong \bigoplus_{i=1}^r \mathbb{N}_{\mathcal{Z}_{\circ,i,a}}$, where $\mathbb{N}_{\mathcal{Z}_{\circ,i,a}}$ is the direct image to $\overline{\mathcal{X}}_{\circ,a}$ of the constant sheaf on $\mathcal{Z}_{\circ,i,a}$ with fiber \mathbb{N} . Then we put $(\overline{\mathcal{X}}_{\circ}, \mathcal{Z}_{\circ})_a^{1/n} := \overline{\mathcal{X}}_{\circ,a} \times_{[\mathbb{A}_{W_a(k)}^r/\mathbb{G}_{m,W_a(k)}^r], n} [\mathbb{A}_{W_a(k)}^r/\mathbb{G}_{m,W_a(k)}^r]$ for $n \in \mathbb{N}$ prime to p , and put $(\overline{\mathcal{X}}_{\circ}, \mathcal{Z}_{\circ})^{1/n} := \varinjlim_a (\overline{\mathcal{X}}_{\circ}, \mathcal{Z}_{\circ})_a^{1/n}$. Then they admit a canonical endomorphism F_0 which lifts the q -th power map on $(\overline{\mathcal{X}}, \mathcal{Z})^{1/n}$ and which is compatible with $(\sigma|_{W(k)})^*$. Let us put $(\overline{\mathcal{X}}, \mathcal{Z})_a^{1/n} := (\overline{\mathcal{X}}, \mathcal{Z})_{\circ,a}^{1/n} \otimes_{W(k)} \mathcal{O}_K$, $(\overline{\mathcal{X}}, \mathcal{Z})^{1/n} := \varinjlim_a (\overline{\mathcal{X}}, \mathcal{Z})_a^{1/n}$. Then it admits the endomorphism $F := F_{\circ} \otimes \sigma^*$. Let $\text{Vect}((\overline{\mathcal{X}}, \mathcal{Z})^{1/n})$ be the category of locally free modules of finite rank on $(\overline{\mathcal{X}}, \mathcal{Z})^{1/n}$ (= a compatible family of locally free modules of finite rank on $(\overline{\mathcal{X}}, \mathcal{Z})_a^{1/n}$) and let $F\text{-Latt}((\overline{\mathcal{X}}, \mathcal{Z})^{1/n})^\circ$ be the category of unit-root F -lattices on $(\overline{\mathcal{X}}, \mathcal{Z})^{1/n}$, that is, the category of pairs (\mathcal{E}, ϕ) consisting of a locally free module \mathcal{E} of finite rank on $(\overline{\mathcal{X}}, \mathcal{Z})^{1/n}$ and an isomorphism $\phi : F^* \mathcal{E} \xrightarrow{\sim} \mathcal{E}$. Then we have the limits

$$\varinjlim_{(n,p)=1} \text{Vect}((\overline{\mathcal{X}}, \mathcal{Z})^{1/n}), \quad \varinjlim_{(n,p)=1} F\text{-Latt}((\overline{\mathcal{X}}, \mathcal{Z})^{1/n})^\circ.$$

Then we have the following theorem:

Theorem 4.1. *Let the notations be as above.*

(1) *There exists an equivalence*

$$(4.5) \quad \text{Rep}_{O_K^\sigma}(\pi_1^t(X)) \xrightarrow{\sim} \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} F\text{-Latt}([\overline{Y}^{\text{sm}}/G_Y])^\circ$$

(where $\text{Rep}_{O_K^\sigma}(\pi_1^t(\overline{X}))$ denotes the category of continuous representations of the tame fundamental group $\pi_1^t(\overline{X})$ (tamely ramified at generic points of Z) to free O_K^σ -modules of finite rank) and an equivalence

$$(4.6) \quad \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} F\text{-Latt}([\overline{Y}^{\text{sm}}/G_Y])_{\mathbb{Q}}^\circ \xrightarrow{\sim} \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} F\text{-Isoc}([\overline{Y}^{\text{sm}}/G_Y])^\circ$$

such that the following diagram is commutative:

$$(4.7) \quad \begin{array}{ccc} \text{Rep}_{O_K^\sigma}(\pi_1^t(X))_{\mathbb{Q}} & \xrightarrow{(4.5)_{\mathbb{Q}}} & \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} F\text{-Latt}([\overline{Y}^{\text{sm}}/G_Y])_{\mathbb{Q}}^\circ \\ \parallel & & \downarrow (4.6) \\ \text{Rep}_{K^\sigma}(\pi_1^t(X)) & \xrightarrow{(2.16)} & \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} F\text{-Isoc}([\overline{Y}^{\text{sm}}/G_Y])^\circ. \end{array}$$

(2) *There exist equivalences*

$$(4.8) \quad \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} F\text{-Latt}([\overline{\mathcal{Y}}^{\text{sm}}/G_Y])^\circ \xrightarrow{=} \varinjlim_{(n,p)=1} F\text{-Latt}((\overline{\mathcal{X}}, \mathcal{Z})^{1/n})^\circ,$$

$$(4.9) \quad \varinjlim_{(n,p)=1} F\text{-Latt}((\overline{\mathcal{X}}, \mathcal{Z})^{1/n})_{\mathbb{Q}}^\circ \xrightarrow{=} \varinjlim_{(n,p)=1} F\text{-Isoc}((\overline{\mathcal{X}}, \mathcal{Z})^{1/n})^\circ$$

which makes the following diagram commutative:

$$(4.10) \quad \begin{array}{ccc} \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} F\text{-Latt}([\overline{\mathcal{Y}}^{\text{sm}}/G_Y])_{\mathbb{Q}}^\circ & \xrightarrow{(4.8)_{\mathbb{Q}}} & \varinjlim_{(n,p)=1} F\text{-Latt}((\overline{\mathcal{X}}, \mathcal{Z})^{1/n})_{\mathbb{Q}}^\circ \\ (4.6) \downarrow & & \downarrow (4.9) \\ \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} F\text{-Isoc}([\overline{\mathcal{Y}}^{\text{sm}}/G_Y])^\circ & \xrightarrow{(2.40)^\circ} & \varinjlim_{(n,p)=1} F\text{-Isoc}((\overline{\mathcal{X}}, \mathcal{Z})^{1/n})^\circ. \end{array}$$

In particular, we have the equivalence

$$(4.11) \quad \text{Rep}_{O_K^\sigma}(\pi_1^t(X)) \xrightarrow{=} \varinjlim_{(n,p)=1} F\text{-Latt}((\overline{\mathcal{X}}, \mathcal{Z})^{1/n})^\circ$$

which is defined as the composite $(4.8) \circ (4.5)$. When X is a curve, we have also a functor

$$(4.12) \quad \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} \text{Vect}([\overline{\mathcal{Y}}^{\text{sm}}/G_Y]) \longrightarrow \varinjlim_{(n,p)=1} \text{Vect}((\overline{\mathcal{X}}, \mathcal{Z})^{1/n}).$$

satisfying $F\text{-(4.12)}^\circ = (4.8)$.

Proof. The method of the proof is similar to that of Theorem 2.18, Proposition 2.23 and Theorem 2.26, as we explain below.

Let $G_Y\text{-Sm}_{O_K^\sigma}(\overline{\mathcal{Y}}^{\text{sm}})$ be the category of smooth O_K^σ -sheaves on $\overline{\mathcal{Y}}^{\text{sm}}$ endowed with equivariant G_Y -action. Then we have the equivalence

$$\text{Rep}_{O_K^\sigma}(\pi_1^t(X)) \xrightarrow{=} \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} G_Y\text{-Sm}_{O_K^\sigma}(\overline{\mathcal{Y}}^{\text{sm}})$$

(which is the integral version of (2.20) and can be proved exactly in the same way). For $m = 0, 1, 2$, let $\overline{\mathcal{Y}}_m^{\text{sm}}, \overline{\mathcal{Y}}_m^{\text{sm}}$ be the $(m+1)$ -fold fiber product of $\overline{\mathcal{Y}}^{\text{sm}}, \overline{\mathcal{Y}}^{\text{sm}}$ over $[\overline{\mathcal{Y}}^{\text{sm}}/G_Y], [\overline{\mathcal{Y}}^{\text{sm}}/G_Y]$ respectively. Then we have 2-truncated simplicial scheme $\overline{\mathcal{Y}}_\bullet^{\text{sm}}$, 2-truncated simplicial formal scheme $\overline{\mathcal{Y}}_\bullet^{\text{sm}}$ and we have the equivalence

$$\begin{aligned} \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} G_Y\text{-Sm}_{O_K^\sigma}(\overline{\mathcal{Y}}^{\text{sm}}) &\xrightarrow{=} \text{Sm}_{O_K^\sigma}(\overline{\mathcal{Y}}_\bullet^{\text{sm}}) \\ &\xrightarrow{(4.3) \text{ for } \overline{\mathcal{Y}}_\bullet^{\text{sm}}} \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} F\text{-Latt}(\overline{\mathcal{Y}}_\bullet^{\text{sm}})^\circ \\ &\xleftarrow{=} \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} F\text{-Latt}([\overline{\mathcal{Y}}^{\text{sm}}/G_Y])^\circ, \end{aligned}$$

where the first and the third equivalence follow from the etale descent. By composing these, we obtain the equivalence (4.5).

Let us denote the category of compatible family of objects in $F\text{-Latt}(\overline{\mathcal{Y}}_m^{\text{sm}})_{\mathbb{Q}}^{\circ}$ ($m = 0, 1, 2$) by $\{F\text{-Latt}(\overline{\mathcal{Y}}_m^{\text{sm}})_{\mathbb{Q}}^{\circ}\}_{m=0,1,2}$. (Note that this is not a priori equal to the \mathbb{Q} -linearization $F\text{-Latt}(\overline{\mathcal{Y}}_{\bullet}^{\text{sm}})_{\mathbb{Q}}^{\circ}$ of the category $F\text{-Latt}(\overline{\mathcal{Y}}_{\bullet}^{\text{sm}})^{\circ}$ of unit-root F -lattices on $\overline{\mathcal{Y}}_{\bullet}^{\text{sm}}$.) Then the functor (4.6) is defined as the composite of equivalences

$$\begin{aligned}
(4.13) \quad \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} F\text{-Latt}([\overline{\mathcal{Y}}^{\text{sm}}/G_Y])_{\mathbb{Q}}^{\circ} &\xrightarrow{=} \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} F\text{-Latt}(\overline{\mathcal{Y}}_{\bullet}^{\text{sm}})_{\mathbb{Q}}^{\circ} \\
&\longrightarrow \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} \{F\text{-Latt}(\overline{\mathcal{Y}}_m^{\text{sm}})_{\mathbb{Q}}^{\circ}\}_{m=0,1,2} \\
&\stackrel{(4.2)}{\xrightarrow{\text{for } \overline{\mathcal{Y}}_{\bullet}^{\text{sm}} =}} \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} F\text{-Isoc}(\overline{\mathcal{Y}}_{\bullet}^{\text{sm}})^{\circ} \\
&\xleftarrow{=} \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} F\text{-Isoc}([\overline{\mathcal{Y}}^{\text{sm}}/G_Y])^{\circ}.
\end{aligned}$$

The commutativity of the diagram (4.7) is the immediate consequence of the construction of the Crew's equivalence G (as the composite (4.2) \circ (4.1)) for $\overline{\mathcal{Y}}_{\bullet}^{\text{sm}}$ and $\overline{\mathcal{Y}}_{\bullet}^{\text{sm}}$. Since (4.5) $_{\mathbb{Q}}$ and (2.16) are equivalences, we see that (4.6) is also an equivalence. So we see that the the functor

$$(4.14) \quad \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} F\text{-Latt}(\overline{\mathcal{Y}}_{\bullet}^{\text{sm}})_{\mathbb{Q}}^{\circ} \longrightarrow \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} \{F\text{-Latt}(\overline{\mathcal{Y}}_m^{\text{sm}})_{\mathbb{Q}}^{\circ}\}_{m=0,1,2}$$

of the second line of (4.13) is an equivalence.

Next, let us recall that the functor $F\text{-Isoc}([\overline{\mathcal{Y}}^{\text{sm}}/G_Y]) \longrightarrow F\text{-Isoc}((\overline{X}, Z)^{1/n})$ is defined as the composite

$$\begin{aligned}
F\text{-Isoc}([\overline{\mathcal{Y}}^{\text{sm}}/G_Y]) &\xrightarrow{=} F\text{-Isoc}(\overline{\mathcal{Y}}_{\bullet}^{\text{sm}}) \\
&\longrightarrow F\text{-Isoc}(\overline{U}_{\bullet\bullet\bullet}) \\
&\xrightarrow{=} F\text{-Isoc}(\overline{X}'_{\bullet\bullet}) \\
&\xrightarrow{=} F\text{-Isoc}(\overline{X}_{\bullet\bullet}) \\
&\xrightarrow{=} F\text{-Isoc}((\overline{X}, Z)^{1/n}),
\end{aligned}$$

where $\overline{\mathcal{Y}}_{\bullet}^{\text{sm}}, \overline{U}_{\bullet\bullet\bullet}, \overline{X}'_{\bullet\bullet}$ are as in the proof of Proposition 2.23. Since they are finite Kummer log etale over its image in (\overline{X}, Z) with respect to suitable log structures associated to certain simple normal crossing divisors, we have the canonical lifting $\overline{\mathcal{Y}}_{\circ,\bullet}^{\text{sm}}, \overline{U}_{\circ,\bullet\bullet\bullet}, \overline{X}'_{\circ,\bullet\bullet}$ of them to p -adic formal schemes smooth over $\text{Spf } W(k)$ and they admit the endomorphism F_{\circ} which lifts the q -th power map on the special fiber and which are compatible with F_{\circ} on $(\overline{\mathcal{X}}_{\circ}, \mathcal{Z}_{\circ})$. If we put $\overline{\mathcal{Y}}_{\bullet}^{\text{sm}} := \overline{\mathcal{Y}}_{\circ,\bullet}^{\text{sm}} \otimes_{W(k)} O_K, \overline{U}_{\bullet\bullet\bullet} := \overline{U}_{\circ,\bullet\bullet\bullet} \otimes_{W(k)} O_K$ and $\overline{X}'_{\bullet\bullet} := \overline{X}'_{\circ,\bullet\bullet} \otimes_{W(k)} O_K$, they are smooth over $\text{Spf } O_K$ and they admit the endomorphism $F := F_{\circ} \otimes \sigma^*$ which lifts the q -th power map on the special

fiber and which are compatible with F on $(\overline{\mathcal{X}}, \mathcal{Z})$. Hence we can define the sequence of functors

$$\begin{aligned}
(4.15) \quad F\text{-Latt}([\overline{\mathcal{Y}}^{\text{sm}}/G_Y])^\circ &\xrightarrow{=} F\text{-Latt}(\overline{\mathcal{Y}}_\bullet^{\text{sm}})^\circ \\
&\longrightarrow F\text{-Latt}(\overline{\mathcal{U}}_{\bullet\bullet\bullet})^\circ \\
&\xrightarrow{=} F\text{-Latt}(\overline{\mathcal{X}}'_{\bullet\bullet})^\circ \\
&\xrightarrow{=} F\text{-Latt}(\overline{\mathcal{X}}_{\bullet\bullet})^\circ \\
&\xrightarrow{=} F\text{-Latt}((\overline{\mathcal{X}}, \mathcal{Z})^{1/n})^\circ
\end{aligned}$$

(where the equality follows from descent property and the fact that $\text{codim}(\overline{\mathcal{X}}_{\bullet\bullet} \setminus \overline{\mathcal{X}}'_{\bullet\bullet}, \overline{\mathcal{X}}_{\bullet\bullet}) \geq 2$). So we have defined the functor (4.8). When X is a curve, we can define the functor (4.12) in the same way as above, noting the equality $\overline{\mathcal{X}} = \overline{\mathcal{X}}'$ in this case.

In the following in this proof, we denote $\overline{\mathcal{X}}_{\bullet\bullet}, \overline{\mathcal{X}}_{\bullet\bullet}$ defined from $(\overline{\mathcal{X}}, Z)^{1/n}, (\overline{\mathcal{X}}, \mathcal{Z})^{1/n}$ by $\overline{\mathcal{X}}_{\bullet\bullet}^{(n)}, \overline{\mathcal{X}}_{\bullet\bullet}^{(n)}$, because they depend on n . Also, let us denote the category of compatible family of objects in $F\text{-Latt}(\overline{\mathcal{X}}_{kl}^{(n)})_{\mathbb{Q}}^\circ$ ($k, l = 0, 1, 2$) by $\{F\text{-Latt}(\overline{\mathcal{X}}_{kl}^{(n)})_{\mathbb{Q}}^\circ\}_{k,l=0,1,2}$. Then the functor (4.9) is defined as the composite

$$\begin{aligned}
(4.16) \quad \varinjlim_{(n,p)=1} F\text{-Latt}((\overline{\mathcal{X}}, \mathcal{Z})^{1/n})_{\mathbb{Q}}^\circ &\xrightarrow{=} \varinjlim_{(n,p)=1} F\text{-Latt}(\overline{\mathcal{X}}_{\bullet\bullet}^{(n)})_{\mathbb{Q}}^\circ \\
&\longrightarrow \varinjlim_{(n,p)=1} \{F\text{-Latt}(\overline{\mathcal{X}}_{kl}^{(n)})_{\mathbb{Q}}^\circ\}_{k,l=0,1,2} \\
&\stackrel{(4.2) \text{ for } \overline{\mathcal{X}}_{\bullet\bullet}^{(n)}}{\longrightarrow} \varinjlim_{(n,p)=1} F\text{-Isoc}(\overline{\mathcal{X}}_{\bullet\bullet}^{(n)})^\circ \\
&\xrightarrow{=} \varinjlim_{(n,p)=1} F\text{-Isoc}((\overline{\mathcal{X}}, Z)^{1/n})^\circ.
\end{aligned}$$

Then one can prove the commutativity (4.10) easily, using the functoriality of (4.2).

Finally we prove that the functor (4.8) is an equivalence. To prove this, it suffices to prove the equivalence (4.11). Note that a part of the functors (4.15)

$$F\text{-Latt}(\overline{\mathcal{Y}}_\bullet^{\text{sm}})^\circ \longrightarrow F\text{-Latt}(\overline{\mathcal{U}}_{\bullet\bullet\bullet})^\circ \xrightarrow{=} F\text{-Latt}(\overline{\mathcal{X}}'_{\bullet\bullet})^\circ \xrightarrow{=} F\text{-Latt}(\overline{\mathcal{X}}_{\bullet\bullet})^\circ$$

is rewritten via the equivalence (4.3) in the following way:

$$(4.17) \quad \text{Sm}_{O_K^\sigma}(\overline{\mathcal{Y}}_\bullet^{\text{sm}}) \longrightarrow \text{Sm}_{O_K^\sigma}(\overline{\mathcal{U}}_{\bullet\bullet\bullet}) \xleftarrow{=} \text{Sm}_{O_K^\sigma}(\overline{\mathcal{X}}'_{\bullet\bullet}) \xleftarrow{=} \text{Sm}_{O_K^\sigma}(\overline{\mathcal{X}}_{\bullet\bullet}).$$

So it induces the functor

$$(4.18) \quad \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} \text{Sm}_{O_K^\sigma}(\overline{\mathcal{Y}}_\bullet^{\text{sm}}) \longrightarrow \varinjlim_{(n,p)=1} \text{Sm}_{O_K^\sigma}(\overline{\mathcal{X}}_{\bullet\bullet}^{(n)})$$

and using this, we can rewrite the functor (4.11) as the composite

$$(4.19) \quad \text{Rep}_{O_K^\sigma}(\pi_1^t(X)) \xrightarrow{=} \varinjlim_{Y \rightarrow X \in \mathcal{G}} \text{Sm}_{O_K^\sigma}(\overline{Y}_\bullet^{\text{sm}}) \xrightarrow{(4.18)} \varinjlim_{(n,p)=1} \text{Sm}_{O_K^\sigma}(\overline{X}_{\bullet\bullet}^{(n)}) \\ \xrightarrow{(4.3)=} \varinjlim_{(n,p)=1} F\text{-Latt}(\overline{\mathcal{X}}_{\bullet\bullet}^{(n)})^\circ \xleftarrow{=} \varinjlim_{(n,p)=1} F\text{-Latt}((\overline{\mathcal{X}}, \mathcal{Z})^{1/n})^\circ.$$

So it suffices to prove that the first line

$$(4.20) \quad \text{Rep}_{O_K^\sigma}(\pi_1^t(X)) \xrightarrow{=} \varinjlim_{Y \rightarrow X \in \mathcal{G}} \text{Sm}_{O_K^\sigma}(\overline{Y}_\bullet^{\text{sm}}) \xrightarrow{(4.18)} \varinjlim_{(n,p)=1} \text{Sm}_{O_K^\sigma}(\overline{X}_{\bullet\bullet}^{(n)})$$

of the functor (4.19) is an equivalence. By construction, the composite of the functor (4.20) and the restriction functor

$$(4.21) \quad \varinjlim_{(n,p)=1} \text{Sm}_{O_K^\sigma}(\overline{X}_{\bullet\bullet}^{(n)}) \longrightarrow \text{Sm}_{O_K^\sigma}(X_{\bullet\bullet})$$

$(X_{\bullet\bullet} := X \times_{\overline{X}} \overline{X}_{\bullet\bullet}^{(n)})$ is equal to the composite $\text{Rep}_{O_K^\sigma}(\pi_1^t(X)) \xrightarrow{\subset} \text{Rep}_{O_K^\sigma}(\pi_1(X)) \xrightarrow{=} \text{Sm}_{O_K^\sigma}(X_{\bullet\bullet})$, which is fully faithful. Also, it is easy to see that (4.21) is faithful. So (2.68) is fully faithful. Also, it is obvious that any object ρ in $\text{Rep}_{O_K^\sigma}(\pi_1(X))$ which is sent to an object in $\varinjlim_{(n,p)=1} \text{Sm}_{O_K^\sigma}(\overline{X}_{\bullet\bullet}^{(n)}) \subset \text{Sm}_{K^\sigma}(X_{\bullet\bullet})$ is tamely ramified along Z . So the functor (4.20) is an equivalence as desired and we are done. (As a corollary, we see that the functor

$$(4.22) \quad \varinjlim_{(n,p)=1} F\text{-Latt}(\overline{\mathcal{X}}_{\bullet\bullet}^{(n)})_{\mathbb{Q}}^\circ \longrightarrow \varinjlim_{(n,p)=1} \{F\text{-Latt}(\overline{\mathcal{X}}_{kl}^{(n)})_{\mathbb{Q}}^\circ\}_{k,l=0,1,2}$$

of the second line in (4.16) is an equivalence.) \square

As for the functor (4.12), we have the following result:

Proposition 4.2. *Let the notations be as above and assume that \overline{X} is a (g, l) -curve, X is a (g, l') -curve ($l' \geq l$). Then the functor (4.12) is an equivalence if $(g, l, l') \neq (0, 0, 1)$ and it is not an equivalence if $(g, l, l') = (0, 0, 1)$.*

Proof. Recall that, in the proof of Theorem 2.28, we have defined the functor

$$\begin{aligned} \text{Isoc}((\overline{X}, Z)^{1/n}) &\xrightarrow{=} \text{Isoc}(\overline{X}_{\bullet\bullet}) \\ &\longrightarrow \text{Isoc}(\widetilde{Y}_{\bullet\bullet\bullet}) \\ &\xrightarrow{=} \text{Isoc}(\overline{Y}_{\bullet\bullet}) \\ &\xrightarrow{=} \text{Isoc}(\overline{Y}_\bullet) = \text{Isoc}([\overline{Y}/G_Y]) \end{aligned}$$

in the case $(g, l, l') \neq (0, 0, 1)$. (The notations are as in there.) In the situation here, we have the natural lifts $\overline{\mathcal{X}}_{\bullet\bullet}, \widetilde{\mathcal{Y}}_{\bullet\bullet\bullet}, \overline{\mathcal{Y}}_{\bullet\bullet}, \overline{\mathcal{Y}}_\bullet$ of $\overline{X}_{\bullet\bullet}, \widetilde{Y}_{\bullet\bullet\bullet}, \overline{Y}_{\bullet\bullet}, \overline{Y}_\bullet$ to p -adic formal

schemes smooth over $\mathrm{Spf} O_K$ (as in the proof of Theorem 4.1) and we can define in the same way the functor

$$\begin{aligned} \mathrm{Vect}((\overline{\mathcal{X}}, \mathcal{Z})^{1/n}) &\xrightarrow{=} \mathrm{Vect}(\overline{\mathcal{X}}_{\bullet\bullet}) \\ &\longrightarrow \mathrm{Vect}(\widetilde{\mathcal{Y}}_{\bullet\bullet\bullet}) \\ &\xrightarrow{=} \mathrm{Vect}(\overline{\mathcal{Y}}_{\bullet\bullet}) \\ &\xrightarrow{=} \mathrm{Vect}(\overline{\mathcal{Y}}_{\bullet}) = \mathrm{Vect}([\overline{\mathcal{Y}}/G_Y]) \end{aligned}$$

inducing the functor

$$\varinjlim_{(n,p)=1} \mathrm{Vect}((\overline{\mathcal{X}}, \mathcal{Z})^{1/n}) \longrightarrow \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} \mathrm{Vect}([\overline{\mathcal{Y}}/G_Y]) = \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} \mathrm{Vect}([\overline{\mathcal{Y}}^{\mathrm{sm}}/G_Y]),$$

which gives the inverse of (4.12).

Let us consider the case $(g, l, l') = (0, 0, 1)$. In this case, we have

$$\varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} \mathrm{Vect}([\overline{\mathcal{Y}}^{\mathrm{sm}}/G_Y]) = \mathrm{Vect}(\overline{\mathcal{X}}).$$

On the other hand, if we define the diagram

$$\begin{array}{ccccc} \mathcal{X} & \xleftarrow{\supset} & \mathcal{V} & \xrightarrow{\subset} & \mathcal{U} \\ & & \uparrow & & \uparrow \tilde{\varphi} \\ & & \mathcal{V}^{(n)} & \xrightarrow{\subset} & \mathcal{U}^{(n)} \end{array}$$

and t as in (2.75) with $n \geq 2$, we have

$$\mathrm{Vect}((\overline{\mathcal{X}}, \mathcal{Z})^{1/n}) = \mathrm{Vect}(\mathcal{X}) \times_{\mathrm{Vect}([\mathcal{V}^{(n)}/\mu_n])} \mathrm{Vect}([\mathcal{U}^{(n)}/\mu_n])$$

(see [3]) and this contains an object of the form $\mathcal{E} = (\mathcal{E}_0, \mathcal{E}_1, \iota)$, where $\mathcal{E}_0 = \mathcal{O}_{\mathcal{X}}$, $\mathcal{E}_1 = t\mathcal{O}_{\mathcal{U}^{(n)}}$ endowed with the action $\zeta \cdot t = \zeta t$ ($\zeta \in \mu_n$) of μ_n and ι is the μ_n -equivariant isomorphism $\mathcal{E}_1|_{\mathcal{V}^{(n)}} = t\mathcal{O}_{\mathcal{V}^{(n)}} \hookrightarrow \mathcal{O}_{\mathcal{V}^{(n)}} = \mathcal{E}_0|_{\mathcal{V}^{(n)}}$. Then the image of \mathcal{E} in the category $\mathrm{Vect}((\overline{\mathcal{X}}, \mathcal{Z})^{1/m})$ for any $n \mid m$ does not come from an object in $\mathrm{Vect}(\overline{\mathcal{X}})$ because $\mathcal{E}_1|_{\mathcal{U}^{(m)}} = t^{m/n}\mathcal{O}_{\mathcal{U}^{(m)}}$ is not locally generated by μ_m -invariant sections. Hence the functor (4.12) is not essentially surjective in this case. \square

Next we define the notion of parabolic vector bundles and parabolic (unit-root) F -lattices on $(\overline{\mathcal{X}}, \mathcal{Z})$.

Definition 4.3. *Let $(\overline{\mathcal{X}}, \mathcal{Z})$ be as above.*

- (1) *A parabolic vector bundle on $(\overline{\mathcal{X}}, \mathcal{Z})$ is an inductive system $(\mathcal{E}_{\alpha})_{\alpha \in \mathbb{Z}_{(p)}^r}$ of vector bundles on $\overline{\mathcal{X}}$ (we denote the transition map by $\iota_{\alpha\beta} : \mathcal{E}_{\alpha} \longrightarrow \mathcal{E}_{\beta}$ for $\alpha = (\alpha_i), \beta = (\beta_i) \in \mathbb{Z}_{(p)}^r$ with $\alpha_i \leq \beta_i$ ($\forall i$)) satisfying the following conditions:*

(a) For any $1 \leq i \leq r$, there is an isomorphism as inductive systems

$$((\mathcal{E}_{\alpha+e_i})_\alpha, (\iota_{\alpha+e_i, \beta+e_i})_{\alpha, \beta}) \cong ((\mathcal{E}_\alpha(\mathcal{Z}_i))_\alpha, (\iota_{\alpha\beta} \otimes \text{id})_{\alpha, \beta})$$

via which the morphism $(\iota_{\alpha, \alpha+e_i})_\alpha : (\mathcal{E}_\alpha)_\alpha \longrightarrow (\mathcal{E}_{\alpha+e_i})_\alpha$ is identified with the morphism $\text{id} \otimes \iota_{0, e_i}^0 : (\mathcal{E}_\alpha)_\alpha \longrightarrow (\mathcal{E}_\alpha(\mathcal{Z}_i))_\alpha$, where $\iota_{0, e_i}^0 : \mathcal{O}_{\overline{\mathcal{X}}} \hookrightarrow \mathcal{O}_{\overline{\mathcal{X}}}(\mathcal{Z}_i)$ denotes the natural inclusion.

(b) There exists a positive integer n prime to p which satisfies the following condition: For any $\alpha = (\alpha_i)_i$, $\iota_{\alpha'\alpha}$ is an isomorphism if we put $\alpha' = ([n\alpha_i]/n)_i$.

(2) A parabolic F -lattice (resp. a parabolic unit-root F -lattice) on $(\overline{\mathcal{X}}, \mathcal{Z})$ is a pair $((\mathcal{E}_\alpha)_{\alpha \in \mathbb{Z}_{(p)}^r}, \Psi := (\Psi_\alpha)_{\alpha \in \mathbb{Z}_{(p)}^r})$ consisting of a parabolic vector bundle $(\mathcal{E}_\alpha)_{\alpha \in \mathbb{Z}_{(p)}^r}$ on $(\overline{\mathcal{X}}, \mathcal{Z})$ endowed with morphisms $\Psi_\alpha : (F^*\mathcal{E}_\alpha)_\mathbb{Q} \longrightarrow \mathcal{E}_{q\alpha, \mathbb{Q}}$ in the category $\text{Coh}(\mathcal{X})_\mathbb{Q}$ (resp. $\Psi_\alpha : F^*\mathcal{E}_\alpha \longrightarrow \mathcal{E}_{q\alpha}$ in the category $\text{Coh}(\mathcal{X})$) such that $\varinjlim_\alpha \Psi_\alpha : \varinjlim_\alpha (F^*\mathcal{E}_\alpha)_\mathbb{Q} \longrightarrow \varinjlim_\alpha \mathcal{E}_{q\alpha, \mathbb{Q}}$ (resp. $\varinjlim_\alpha \Psi_\alpha : \varinjlim_\alpha F^*\mathcal{E}_\alpha \longrightarrow \varinjlim_\alpha \mathcal{E}_{q\alpha}$) is isomorphic as ind-objects.

For $\alpha := (\alpha_i)_i \in \mathbb{Z}_{(p)}^r$, let $\mathcal{O}_{\overline{\mathcal{X}}}(\sum_i \alpha_i \mathcal{Z}_i) := (\mathcal{O}_{\overline{\mathcal{X}}}(\sum_i \alpha_i \mathcal{Z}_i)_\beta)_\beta$ be the parabolic vector bundle on $(\overline{\mathcal{X}}, \mathcal{Z})$ defined by $\mathcal{O}_{\overline{\mathcal{X}}}(\sum_i \alpha_i \mathcal{Z}_i)_\beta := \mathcal{O}_{\overline{\mathcal{X}}}(\sum_i [\alpha_i + \beta_i] \mathcal{Z}_i)$, where $\beta = (\beta_i)_i$. Using this, we define the notion of locally abelian parabolic vector bundles and locally abelian parabolic (unit-root) F -lattices on $(\overline{\mathcal{X}}, \mathcal{Z})$. (This terminology is essentially due to Iyer-Simpson [14].)

Definition 4.4. A parabolic vector bundle $(\mathcal{E}_\alpha)_\alpha$ on $(\overline{\mathcal{X}}, \mathcal{Z})$ is called *locally abelian* if there exists a positive integer n prime to p such that, Zariski locally on $\overline{\mathcal{X}}_n := \overline{\mathcal{X}} \otimes_{O_K} O_K[\mu_n]$, $(\mathcal{E}_\alpha)_\alpha|_{\overline{\mathcal{X}}_n}$ has the form $\bigoplus_{j=1}^\mu \mathcal{O}_{\overline{\mathcal{X}}}(\sum_i \alpha_{ij} \mathcal{Z}_i)|_{\overline{\mathcal{X}}_n}$ for some $\alpha_{ij} \in \frac{1}{n}\mathbb{Z}$ ($1 \leq i \leq r, 1 \leq j \leq \mu$). A parabolic (unit-root) F -lattice $((\mathcal{E}_\alpha)_\alpha, \Psi)$ is called *locally abelian* if so is $(\mathcal{E}_\alpha)_\alpha$.

We denote the category of locally abelian parabolic vector bundles on $(\overline{\mathcal{X}}, \mathcal{Z})$ by $\text{Par-Vect}(\overline{\mathcal{X}}, \mathcal{Z})$, the category of locally abelian parabolic F -lattices on $(\overline{\mathcal{X}}, \mathcal{Z})$ by $\text{Par-}F\text{-Latt}(\overline{\mathcal{X}}, \mathcal{Z})$ and the category of locally abelian parabolic unit-root F -lattices on $(\overline{\mathcal{X}}, \mathcal{Z})$ by $\text{Par-}F\text{-Latt}(\overline{\mathcal{X}}, \mathcal{Z})^\circ$.

Then we have the following theorem.

Theorem 4.5. Let $(\overline{\mathcal{X}}, \mathcal{Z})$ be as above. Then there exist equivalences of categories

$$(4.23) \quad \mathbf{a} : \varinjlim_{(n,p)=1} \text{Vect}((\overline{\mathcal{X}}, \mathcal{Z})^{1/n}) \xrightarrow{\cong} \text{Par-Vect}(\overline{\mathcal{X}}, \mathcal{Z}),$$

$$(4.24) \quad \varinjlim_{(n,p)=1} F\text{-Latt}((\overline{\mathcal{X}}, \mathcal{Z})^{1/n})^\circ \xrightarrow{\cong} \text{Par-}F\text{-Latt}(\overline{\mathcal{X}}, \mathcal{Z})^\circ,$$

$$(4.25) \quad \varinjlim_{(n,p)=1} F\text{-Latt}((\overline{\mathcal{X}}, \mathcal{Z})^{1/n}) \xrightarrow{\cong} \text{Par-}F\text{-Latt}(\overline{\mathcal{X}}, \mathcal{Z}).$$

In particular, we have the equivalence (0.13) defined as the composite

$$\mathrm{Rep}_{\mathcal{O}_K^\circ}(\pi_1^t(X)) \xrightarrow{(4.11)} \varinjlim_{(n,p)=1} F\text{-Latt}((\overline{\mathcal{X}}, \mathcal{Z})^{1/n})^\circ \xrightarrow{(4.24)} \mathrm{Par}\text{-}F\text{-Latt}(\overline{\mathcal{X}}, \mathcal{Z})^\circ.$$

Proof. The proof of the equivalence (4.23) we give below is essentially due to Iyer-Simpson [14]. (See also Borne [3], [4].)

Let us fix $n \in \mathbb{N}$ which is prime to p . Then, we have the inductive system of line bundles

$$(\mathcal{O}(\sum_i \alpha_i \mathcal{Z}_i))_{\alpha=(\alpha_i)_{i \in (\frac{1}{n}\mathbb{Z}/\mathbb{Z})^r}}$$

on $(\overline{\mathcal{X}}, \mathcal{Z})^{1/n}$ (see [14, p.353]). (Here we give a definition using log structure: It suffices to define the inductive system of line bundles $(\mathcal{O}(\sum_i \alpha_i \mathcal{Z}_i)_a)_{\alpha=(\alpha_i)_{i \in (\frac{1}{n}\mathbb{Z}/\mathbb{Z})^r}}$ on $(\overline{\mathcal{X}}, \mathcal{Z})_a^{1/n}$ ($a \in \mathbb{N}$) which are compatible with respect to a . Let $Y \xrightarrow{g} (\overline{\mathcal{X}}, \mathcal{Z})_a^{1/n}$ be a surjective etale morphism from some scheme Y . Then the composite $Y \xrightarrow{g} (\overline{\mathcal{X}}, \mathcal{Z})_a^{1/n} \xrightarrow{\mathrm{proj.}} (\overline{\mathcal{X}}_\circ, \mathcal{Z}_\circ)_a^{1/n} \xrightarrow{\mathrm{proj.}} [\mathbb{A}_{W_a(k)}^r / \mathbb{G}_{m, W_a(k)}^r]$ corresponds to a log structure $M \xrightarrow{\psi} \mathcal{O}_Y$ on Y and a morphism $\gamma : \mathbb{N}^r \longrightarrow \overline{M} := M / \mathcal{O}_Y^\times$ which is liftable to a chart etale locally. Take Y etale local enough so that γ is lifted to a chart $\tilde{\gamma} : \mathbb{N}^r \longrightarrow M$. Let us put $Y' := Y \times_{(\overline{\mathcal{X}}, \mathcal{Z})_a^{1/n}} Y$ and denote the j -th projection $Y' \longrightarrow Y$ by π_j . Then, for $\alpha := (\alpha_i)_i \in (\frac{1}{n}\mathbb{Z}/\mathbb{Z})^r$, take $\alpha_+ := (\alpha_{+,i})_i, \alpha_- := (\alpha_{-,i})_i$ in $(\frac{1}{n}\mathbb{Z}/\mathbb{Z})^r$ with $\alpha = \alpha_+ - \alpha_-$, $\alpha_{+,i}, \alpha_{-,i} \geq 0$ ($\forall i$) and define the line bundle $\mathcal{O}(\sum_i \alpha_i \mathcal{Z}_i)_a$ by patching the trivial line bundle \mathcal{O}_Y on Y by the isomorphism $\pi_2^* \mathcal{O}_Y \xrightarrow{=} \pi_1^* \mathcal{O}_Y$ defined by the section $u \in \Gamma(Y', \mathcal{O}_{Y'}^\times)$ satisfying $u\pi_1^*(\tilde{\gamma}(n\alpha_-))\pi_2^*(\tilde{\gamma}(n\alpha_+)) = \pi_1^*(\tilde{\gamma}(n\alpha_+))\pi_2^*(\tilde{\gamma}(n\alpha_-))$. For $\alpha = (\alpha_i)_i, \beta := (\beta_i)_i$ in $(\frac{1}{n}\mathbb{Z}/\mathbb{Z})^r$ with $\alpha_i \leq \beta_i$ ($\forall i$), the homomorphism $\mathcal{O}(\sum_i \alpha_i \mathcal{Z}_i)_a \longrightarrow \mathcal{O}(\sum_i \beta_i \mathcal{Z}_i)_a$ is defined as the multiplication by $\psi \circ \tilde{\gamma}(n(\beta - \alpha))$ on Y .)

For an object \mathcal{E} in $\mathrm{Vect}((\overline{\mathcal{X}}, \mathcal{Z})^{1/n})$ and $\alpha = (\alpha_i)_i \in \mathbb{Z}_{(p)}^r$, let us define $\mathbf{a}(\mathcal{E})_\alpha := \pi_*(\mathcal{E} \otimes \mathcal{O}_{\overline{\mathcal{X}}}(\sum_i ([n\alpha_i]/n) \mathcal{Z}_i))$, where π denotes the morphism $(\overline{\mathcal{X}}, \mathcal{Z})^{1/n} \longrightarrow \overline{\mathcal{X}}$. Then $\mathbf{a}(\mathcal{E}) := (\mathbf{a}(\mathcal{E})_\alpha)_\alpha$ forms an inductive system of sheaves on $\overline{\mathcal{X}}$.

Let us examine the local structure of $\mathbf{a}(\mathcal{E})$. Let us take an affine open formal subscheme $\mathcal{U} = \mathrm{Spf} R \subseteq \overline{\mathcal{X}}_n := \overline{\mathcal{X}} \otimes_{\mathcal{O}_K} \mathcal{O}_K[\mu_n]$ such that $\mathcal{U} \times_{\overline{\mathcal{X}}} \mathcal{Z}_i$ is defined as $\{t_i = 0\}$ for some t_i ($1 \leq i \leq r$), and let us put $\mathcal{U}^{(n)} := \mathrm{Spf} R[s_i]_{1 \leq i \leq r} / (s_i^n - t_i)_{1 \leq i \leq r}$, $\mathcal{Z}_i^{(n)} := \{s_i = 0\}$ and let us denote the natural morphism $\mathcal{U}^{(n)} \longrightarrow \mathcal{U}$ by $\pi^{(n)}$. Then $G = \mu_n^r \cong (\mathbb{Z}/n\mathbb{Z})^r$ naturally acts on $\mathcal{U}^{(n)}$ (as the action on s_i 's). Let us take a closed point x of \mathcal{U} and put $x^{(n)} := (\mathcal{U}^{(n)} \times_{\mathcal{U}} x)_{\mathrm{red}}$. Then G acts naturally on $x^{(n)}$. For a character $\xi : G \longrightarrow \mu_n$, let $\mathcal{O}_{x^{(n)}}(\xi)$ be the structure sheaf $\mathcal{O}_{x^{(n)}}$ on $x^{(n)}$ endowed with the equivariant G -action by which $g \in G$ acts as $g^* \mathcal{O}_{x^{(n)}} \xrightarrow{g^*} \mathcal{O}_{x^{(n)}} \xrightarrow{\xi(g)} \mathcal{O}_{x^{(n)}}$. Then it is easy to see that the restriction $\mathcal{E}|_{x^{(n)}}$ of \mathcal{E} to $x^{(n)}$ has the form $\bigoplus_{j=1}^\mu \mathcal{O}_{x^{(n)}}(\xi_j)$ for some characters $\xi_j : G \longrightarrow \mu_n$ ($1 \leq j \leq \mu$). Note that, for a character $\xi : G \longrightarrow \mu_n$, we can define the sheaf $\mathcal{O}_{\mathcal{U}^{(n)}}(\xi)$ on $\mathcal{U}^{(n)}$ endowed with an equivariant G -action in the same way as $\mathcal{O}_{x^{(n)}}(\xi)$. Now let us put $\mathcal{F} := \bigoplus_{j=1}^\mu \mathcal{O}_{\mathcal{U}^{(n)}}(\xi_j)$ and let $u_0 : \mathcal{F} \longrightarrow \mathcal{E}|_{\mathcal{U}^{(n)}}$ be any $\mathcal{O}_{\mathcal{U}^{(n)}}$ -linear homomorphism which lifts the canonical isomorphism $\mathcal{F}|_{\overline{\mathcal{X}}^{(n)}} \cong$

$\mathcal{E}|_{\overline{\mathcal{X}}^{(n)}}$. Then $u := \sum_{g \in G} g^{-1} u_0 g$ gives a G -equivariant $\mathcal{O}_{\mathcal{U}^{(n)}}$ -linear homomorphism lifting the isomorphism $\mathcal{F}|_{\overline{\mathcal{X}}^{(n)}} \cong \mathcal{E}|_{\overline{\mathcal{X}}^{(n)}}$. Hence there exists an element $f_x \in \Gamma(\mathcal{U}, \mathcal{O}_{\mathcal{U}})$ with $x \in \{f_x \neq 0\} =: \mathcal{U}_x \subseteq \mathcal{U}$ such that u is isomorphic on $\mathcal{U}_x^{(n)} := \pi^{(n), -1}(\mathcal{U}_x)$. Then we have

$$\begin{aligned} & (\mathbf{a}(\mathcal{E})_\alpha)_\alpha|_{\mathcal{U}_x} \\ &:= (\pi_*(\mathcal{E} \otimes \mathcal{O}(\sum_i [n\alpha_i]/n\mathcal{Z}_i)))_\alpha|_{\mathcal{U}_x} = (\pi_*^{(n)}(\mathcal{E}_{\mathcal{U}^{(n)}} \otimes \mathcal{O}_{\mathcal{U}^{(n)}}(\sum_i [n\alpha_i]\mathcal{Z}_i^{(n)}))^G)_\alpha|_{\mathcal{U}_x} \\ &\cong (\pi_*^{(n)}(\mathcal{F} \otimes \mathcal{O}_{\mathcal{U}^{(n)}}(\sum_i [n\alpha_i]\mathcal{Z}_i^{(n)}))^G)_\alpha|_{\mathcal{U}_x} = \oplus_{j=1}^\mu (\pi_*^{(n)}(\mathcal{O}_{\mathcal{U}^{(n)}}(\xi_j)(\sum_i [n\alpha_i]\mathcal{Z}_i^{(n)}))^G)_\alpha|_{\mathcal{U}_x} \end{aligned}$$

and we can check that there exists some $a_j = (a_{ji})_i \in (\frac{1}{n}\mathbb{Z})^r$ ($1 \leq j \leq \mu$) such that the above inductive system is isomorphic to

$$\oplus_{j=1}^\mu (\mathcal{O}_{\overline{\mathcal{X}}}(\sum_i [a_{ji} + [n\alpha_i]/n]\mathcal{Z}_i))_\alpha|_{\mathcal{U}_x} = \oplus_{j=1}^\mu (\mathcal{O}_{\overline{\mathcal{X}}}(\sum_i [a_{ji} + \alpha_i]\mathcal{Z}_i))_\alpha|_{\mathcal{U}_x}.$$

Hence $(\mathbf{a}(\mathcal{E})_\alpha)_\alpha$ is a locally abelian parabolic vector bundle on $(\overline{\mathcal{X}}, \mathcal{Z})$. The full faithfulness of the functor \mathbf{a} can be checked locally and hence we may check it for the objects of the form $\mathcal{F} = \oplus_{j=1}^\mu \mathcal{O}_{\mathcal{U}^{(n)}}(\xi_j)$, and it is easy in this case. The essential surjectivity can be also checked locally and so it is enough to check it for the objects of the form $\oplus_{j=1}^\mu \mathcal{O}_{\overline{\mathcal{X}}}(\sum_i \alpha_{ij}\mathcal{Z}_i)$, which can be easily seen. So we have proved the equivalence (4.23).

We give an explicit quasi-inverse functor

$$\mathbf{b} : \text{Par-Vect}(\overline{\mathcal{X}}, \mathcal{Z}) \longrightarrow \varinjlim_{(n,p)=1} \text{Vect}((\overline{\mathcal{X}}, \mathcal{Z})^{1/n})$$

in the following way, as in [3], [4]: For an object $\mathcal{E} := (\mathcal{E}_\alpha)_\alpha$ in $\text{Par-Vect}(\overline{\mathcal{X}}, \mathcal{Z})$, take $n \in \mathbb{N}$ prime to p as in Definition 4.3 (1)(b) and let $\mathbf{b}(\mathcal{E})$ be the coend of the family $\{\mathcal{O}(\sum_i -a_i\mathcal{Z}_i) \otimes \pi^*\mathcal{E}_b\}_{a=(a_i)_i, b \in (\frac{1}{n}\mathbb{Z})^r}$, that is, the universal object in $\text{Vect}((\overline{\mathcal{X}}, \mathcal{Z})^{1/n})$ which admits morphisms

$$f_a : \mathcal{O}(\sum_i -a_i\mathcal{Z}_i) \otimes \pi^*\mathcal{E}_a \longrightarrow \mathbf{b}(\mathcal{E})$$

for any $a = (a_i)_i \in (\frac{1}{n}\mathbb{Z})^r$ making the diagram

$$\begin{array}{ccc} \mathcal{O}(\sum_i -a_i\mathcal{Z}_i) \otimes \pi^*\mathcal{E}_b & \longrightarrow & \mathcal{O}(\sum_i -a_i\mathcal{Z}_i) \otimes \pi^*\mathcal{E}_a \\ \downarrow & & \downarrow f_a \\ \mathcal{O}(\sum_i -b_i\mathcal{Z}_i) \otimes \pi^*\mathcal{E}_b & \xrightarrow{f_b} & \mathbf{b}(\mathcal{E}) \end{array}$$

commutative for any $a = (a_i)_i, b = (b_i)_i \in (\frac{1}{n}\mathbb{Z})^r$ with $a_i \geq b_i$ ($\forall i$): The existence of such object and the base change property for flat morphism are checked rather easily

for the objects of the form $\bigoplus_{j=1}^{\mu} \mathcal{O}_{\overline{\mathcal{X}}}(\sum_i \alpha_{ij} Z_i)$ (see [3, 3.17]), and this implies the existence in general case by descent. We can construct the morphism $\mathbf{a} \circ \mathbf{b} \rightarrow \text{id}$ in the same way as [3, 3.18] and check that this is an isomorphism by looking at locally and arguing as [3, 3.17]. By looking at the proof in [3, 3.17], we see the following two facts: $\mathbf{b}(\mathcal{E})$ is in fact the coend of the family $\{\mathcal{O}(\sum_i -a_i Z_i) \otimes \pi^* \mathcal{E}_b\}_{a=(a_i)_i, b \in (\frac{1}{n}\mathbb{Z} \cap [0,1))^r}$, since so is it when \mathcal{E} has the form $\bigoplus_{j=1}^{\mu} \mathcal{O}_{\overline{\mathcal{X}}}(\sum_i \alpha_{ij} Z_i)$. Also, we see that, when \mathcal{E} has the form $\mathcal{O}_{\overline{\mathcal{X}}}(\sum_i \alpha_i Z_i)$, there exists some $a \in (\frac{1}{n}\mathbb{Z} \cap [0,1))^r$ such that f_a is an isomorphism. By the former fact, there exists the canonical map

$$(4.26) \quad \mathbf{b}(\mathcal{E}) \longrightarrow \mathcal{O} \otimes \pi^* \mathcal{E}_1 = \pi^* \mathcal{E}_1$$

and it is injective because, when \mathcal{E} has the form $\mathcal{O}_{\overline{\mathcal{X}}}(\sum_i \alpha_i Z_i)$, the first map in (4.26) is identified with the map $\mathcal{O}(-\sum_i a_i Z_i) \otimes \pi^* \mathcal{E}_a \hookrightarrow \mathcal{O} \otimes \pi^* \mathcal{E}_1$ for some $a \in (\frac{1}{n}\mathbb{Z} \cap [0,1))^r$ by the latter fact above.

Now we define the functors (4.24), (4.25). Let (\mathcal{E}, Ψ) be an object in $F\text{-Latt}((\overline{\mathcal{X}}, \mathcal{Z})^{1/n})^\circ$. Then we define the morphisms $\Psi_\alpha : F^* \mathbf{a}(\mathcal{E})_\alpha \rightarrow \mathbf{a}(\mathcal{E})_{q\alpha}$ as the composite

$$(4.27) \quad \begin{aligned} F^* \mathbf{a}(\mathcal{E})_\alpha &= F^* \pi_*(\mathcal{E} \otimes \mathcal{O}_{\overline{\mathcal{X}}}(\sum_i ([n\alpha_i]/n) Z_i)) \\ &\longrightarrow \pi_* F^*(\mathcal{E} \otimes \mathcal{O}_{\overline{\mathcal{X}}}(\sum_i ([n\alpha_i]/n) Z_i)) \\ &\xrightarrow{\Psi} \pi_*(\mathcal{E} \otimes \mathcal{O}_{\overline{\mathcal{X}}}(\sum_i (q[n\alpha_i]/n) Z_i)) \\ &\longrightarrow \pi_*(\mathcal{E} \otimes \mathcal{O}_{\overline{\mathcal{X}}}(\sum_i ([nq\alpha_i]/n) Z_i)) = \mathbf{a}(\mathcal{E})_{q\alpha}. \end{aligned}$$

When (\mathcal{E}, Ψ) is an object in $F\text{-Latt}((\overline{\mathcal{X}}, \mathcal{Z})^{1/n})$, we can define the morphisms $\Psi_\alpha : (F^* \mathbf{a}(\mathcal{E})_\alpha)_{\mathbb{Q}} \rightarrow (\mathbf{a}(\mathcal{E})_{q\alpha})_{\mathbb{Q}}$ in the same way.

Let \mathcal{U}, t_i be as above. We prove that the map $\Psi_\alpha|_{\mathcal{U}}$ is injective and the cokernel of it is killed by some power of $t^1 := \prod_{i=1}^r t_i$. To see this, it suffices to prove the same property for the first arrow in (4.27), and we are reduced to showing the same property for the map

$$F^* \pi_*^{(n)}(\mathcal{O}_{\mathcal{U}^{(n)}}(\sum_i [n\alpha_i] \mathcal{Z}_i^{(n)})) \longrightarrow \pi_*^{(n)} F^*(\mathcal{O}_{\mathcal{U}^{(n)}}(\sum_i [n\alpha_i] \mathcal{Z}_i^{(n)})).$$

Let us put $A := R[s_i]_{1 \leq i \leq r} / (s_i^n - t_i)_{1 \leq i \leq r}$, $c := \prod_{i=1}^r s_i^{-[n\alpha_i]} \in \text{Frac } A$. Then the above map is rewritten as

$$(4.28) \quad R \otimes_{F^*, R} cA \longrightarrow F^*(c)A; \quad r \otimes x \mapsto rF^*(x),$$

where $F^* : R \rightarrow R, F^* : A \rightarrow A$ are the homomorphisms induced by F on $(\mathcal{U}, \mathcal{Z} \cap \mathcal{U})$ and $(\mathcal{U}^{(n)}, \cup_{i=1}^r \mathcal{Z}_i^{(n)})$. Let us first prove the injectivity of the map (4.28). By assumption on F , we can write $F^*(s_i) = s_i^q u_i$ for some $u_i \in 1 +$

$\mathfrak{m}_K A (1 \leq i \leq r)$. $1 \otimes cs^m$ for $m = (m_i)_{1 \leq i \leq r} \in \{0, \dots, n-1\}^r$ forms a basis of $R \otimes_{F^*, R} cA$ as R -module and they are sent to $F^*(c)s^{qm}u^m$. Noting the fact that qm 's are mutually different modulo n and the fact $u^m \in 1 + \mathfrak{m}_K A$, we see that $F^*(c)s^{qm}u^m$'s are linearly independent over R . Hence the map (4.28) is injective. Prove now that the cokernel of (4.28) is killed by a power of t^1 . For $j = (j_i) \in \{0, \dots, q-1\}^r$, choose $N(j) = (N(j)_i)_i, M(j) = (M(j)_i)_i \in \mathbb{N}^r$ such that $j + nN(j) = qM(j)$. Then $\{F^*(c)s^j u^{M(j)}\}_j$ generates $F^*(c)A$ over $RF^*(A)$, where $RF^*(A)$ denotes the R -subalgebra of A generated by $F^*(A)$. So it suffices to prove that the image of each $F^*(c)s^j u^{M(j)}$ in $\text{Coker}(4.28)$ is killed by a power of t^1 . Noting that $t^{N(j)}F^*(c)s^j u^{M(j)} = F^*(c)(s^{qM(j)}u^{M(j)}) = F^*(cs^{M(j)})$ is in the image of (4.28), we see that the image of each $F^*(c)s^j u^{M(j)}$ in $\text{Coker}(4.28)$ is killed by $(t^1)^{\max_{j,i} N(j)_i}$ and we have the desired property.

Now we prove the claim that $\varinjlim_{\alpha} \Psi_{\alpha}$ is an isomorphism as morphism between ind-objects. Here we will work only in the case $\mathcal{E} \in \varinjlim_{(n,p)=1} F\text{-Latt}((\overline{\mathcal{X}}, \mathcal{Z})^{1/n})^{\circ}$, because the proof in the case $\mathcal{E} \in \varinjlim_{(n,p)=1} F\text{-Latt}((\overline{\mathcal{X}}, \mathcal{Z})^{1/n})$ can be done exactly in the same way. To prove the claim, it suffices to define the morphism $\Psi' : \varinjlim_{\alpha} \mathbf{a}(\mathcal{E})_{\alpha} \longrightarrow \varinjlim_{\alpha} F^* \mathbf{a}(\mathcal{E})_{\alpha}$ which is inverse to $\varinjlim_{\alpha} \Psi_{\alpha}$. By the injectivity of Ψ_{α} 's proven above, it suffices to work locally. Let us take $x \in \mathbf{a}(\mathcal{E})_{\alpha}$ with $\alpha = (\alpha_i)_i$ with $\alpha_i \geq 0$. Then there exists some $N \in \mathbb{N}^r$ (which depends only on α) such that $(t^1)^N x$, regarded as an element of $\mathbf{a}(\mathcal{E})_{q\alpha}$, is in the image of Ψ_{α} . If we take $M, L \in \mathbb{N}$ such that $qM - N = L$, we see that the element $u(t^1)^{-L}x$ in $\mathbf{a}(\mathcal{E})_{q(\alpha+M\mathbf{1})}$ for some $u \in 1 + \mathfrak{m}_K \mathcal{O}_{\overline{\mathcal{X}}}$ can be written as $\Psi_{\alpha+M\mathbf{1}}(y)$ for some $y \in F^* \mathbf{a}(\mathcal{E})_{\alpha+M\mathbf{1}}$. Then we can define Ψ' by $\Psi'(x) := u^{-1}t^L \otimes y \in F^* \mathbf{a}(\mathcal{E})_{\alpha+M\mathbf{1}} \hookrightarrow \varinjlim_{\alpha} F^* \mathbf{a}(\mathcal{E})_{\alpha}$. It is easy to check that this Ψ' is the inverse of $\varinjlim_{\alpha} \Psi_{\alpha}$. Therefore $\varinjlim_{\alpha} \Psi_{\alpha}$ is an isomorphism as morphism between ind-objects, as desired. So $(\mathbf{a}(\mathcal{E}), (\Psi_{\alpha})_{\alpha})$ defines an object in $\text{Par-}F\text{-Latt}(\overline{\mathcal{X}}, \mathcal{Z})^{(\circ)}$ and hence we have defined the functors (4.24), (4.25).

Next we define the functors

$$(4.29) \quad \text{Par-}F\text{-Latt}(\overline{\mathcal{X}}, \mathcal{Z})^{\circ} \longrightarrow \varinjlim_{(n,p)=1} F\text{-Latt}((\overline{\mathcal{X}}, \mathcal{Z})^{1/n})^{\circ},$$

$$(4.30) \quad \text{Par-}F\text{-Latt}(\overline{\mathcal{X}}, \mathcal{Z}) \longrightarrow \varinjlim_{(n,p)=1} F\text{-Latt}((\overline{\mathcal{X}}, \mathcal{Z})^{1/n}),$$

of the converse direction. Since the construction is the same, we explain only the construction of (4.29). Let $(\mathcal{E} := (\mathcal{E}_{\alpha})_{\alpha}, (\Psi_{\alpha})_{\alpha})$ be an object in $\text{Par-}F\text{-Latt}(\overline{\mathcal{X}}, \mathcal{Z})^{\circ}$. Then the maps

$$(4.31) \quad \begin{aligned} F^* \mathcal{O}(\sum_i -a_i \mathcal{Z}_i) \otimes F^* \pi^* \mathcal{E}_b &\longrightarrow \mathcal{O}(\sum_i -qa_i \mathcal{Z}_i) \otimes \pi^* F^* \mathcal{E}_b \\ &\longrightarrow \mathcal{O}(\sum_i -qa_i \mathcal{Z}_i) \otimes \pi^* \mathcal{E}_{qb} \end{aligned}$$

induced by Ψ_b induces, by taking coend, the morphism $\Psi : F^*\mathbf{b}(\mathcal{E}) \longrightarrow \mathbf{b}(\mathcal{E})$. (4.26) induces the commutative diagram

$$\begin{array}{ccc} F^*\mathbf{b}(\mathcal{E}) & \xrightarrow{F^*(4.26)} & F^*\pi^*\mathcal{E}_1 \\ \Psi \downarrow & & \Psi_1 \downarrow \\ \mathbf{b}(\mathcal{E}) & \longrightarrow & \pi^*\mathcal{E}_q \end{array}$$

(where the lower horizontal arrow is the composite $\mathbf{b}(\mathcal{E}) \xrightarrow{(4.26)} \pi^*\mathcal{E}_1 \xrightarrow{\subset} \pi^*\mathcal{E}_q$) and since the horizontal arrows and Ψ_1 are injective, Ψ is also injective. Let us prove the surjectivity of Ψ . To do so, we may work locally. Let us take any $x_0 \in \mathbf{b}(\mathcal{E})$. When \mathcal{E} has the form $\mathcal{O}(\sum_i \alpha_i \mathcal{Z}_i)$, x_0 is a sum of elements of the form $f_c(h \otimes \pi^*x)$ for some fixed $c \in (\frac{1}{n}\mathbb{Z}/\mathbb{Z})^r$ since we can take f_c to be an isomorphism, and since we have $f_c(h \otimes \pi^*x) = f_{c+m}(ht^m \otimes \pi^*t^{-m}x)$ for $m \in \mathbb{Z}^r$, we can change c in order that c has the form qa . In general case, we see from this observation that x_0 is a sum of elements of the form $f_{qa}(h \otimes \pi^*x)$ (a also varies this time) etale locally. So, to prove the surjectivity, we may assume that $x_0 = f_{qa}(h \otimes \pi^*x)$. Then there exists some $N \in \mathbb{N}^r$ such that $(t^1)^N x$ is in the image of Ψ_a . If we take $M, L \in \mathbb{N}$ such that $pM - N = L$, we see that the element $u(t^1)^{-L}x$ in $\mathcal{E}_{q(a+M\mathbf{1})}$ for some $u \in 1 + \mathbf{m}_K \mathcal{O}_{\overline{\mathcal{X}}}$ can be written as $\Psi_{a+M\mathbf{1}}(y)$ for some $y \in F^*\mathcal{E}_{a+M\mathbf{1}}$. Then x_0 is equal to the image of $u^{-1}(t^1)^L h \otimes \pi^*(u(t^1)^{-L}x) \in \mathcal{O}(\sum_i -q(a_i + M)\mathcal{Z}_i) \otimes \pi^*\mathcal{E}_{q(a+M\mathbf{1})}$ which is in the image of Ψ , by definition of it. So Ψ is an isomorphism and so $(\mathbf{b}(\mathcal{E}), \Psi)$ defines an object in $F\text{-Latt}((\overline{\mathcal{X}}, \mathcal{Z})^{1/n})^\circ$.

We see that the functor (4.29) (resp. (4.30)) is the inverse of the functor (4.24) (resp. (4.25)) using the fact that the (parabolic unit-root) F -lattice structure is determined by the underlying structure on (parabolic) vector bundle structure and the morphism Ψ on $\mathcal{X} = \overline{\mathcal{X}} \setminus \mathcal{Z}$. So we have shown that the functors (4.24), (4.25) are equivalent and hence we are done. \square

Remark 4.6. By Corollary 3.7, Theorem 4.1 and Theorem 4.5, we have the equivalence

$$\begin{aligned} (4.32) \quad \text{Par-}F\text{-Latt}(\overline{\mathcal{X}}, \mathcal{Z})_{\mathbb{Q}}^\circ &\xrightarrow{\cong} \varinjlim_{(n,p)=1} F\text{-Latt}((\overline{\mathcal{X}}, \mathcal{Z})^{1/n})_{\mathbb{Q}}^\circ \\ &\xrightarrow{\cong} \varinjlim_{(n,p)=1} F\text{-Isoc}((\overline{\mathcal{X}}, \mathcal{Z})^{1/n})^\circ \\ &\xrightarrow{\cong} \text{Par-}F\text{-Isoc}^{\log}(\overline{\mathcal{X}}, \mathcal{Z})_{\mathbf{0-ss}}^\circ, \end{aligned}$$

which is a parabolic version of (4.2). Therefore, an object $((\mathcal{E}_\alpha)_\alpha, (\Psi_\alpha)_\alpha)_{\mathbb{Q}}$ in the category $\text{Par-}F\text{-Latt}(\overline{\mathcal{X}}, \mathcal{Z})_{\mathbb{Q}}^\circ$ (here, for an object A in an additive category \mathcal{C} , $A_{\mathbb{Q}}$ denotes the object A regarded as an object in $\mathcal{C}_{\mathbb{Q}}$) is sent to an object in $\text{Par-}F\text{-Isoc}^{\log}(\overline{\mathcal{X}}, \mathcal{Z})_{\mathbf{0-ss}}^\circ$, which induces an inductive system of log- ∇ -modules $(\mathcal{E}'_\alpha, \nabla'_\alpha)_\alpha$ on $(\overline{\mathcal{X}}_K, \mathcal{Z}_K)$ endowed with horizontal isomorphism $\Psi' : \varinjlim_\alpha F^*\mathcal{E}'_\alpha \longrightarrow \varinjlim_\alpha \mathcal{E}'_\alpha$ of ind-objects.

Note that the \mathbb{Q} -linearization of the category of coherent sheaves on $\overline{\mathcal{X}}$ is equivalent to the category of coherent sheaves on $\overline{\mathcal{X}}_K$. We prove in this remark that, in these equivalent categories, $(\mathcal{E}_\alpha)_{\alpha, \mathbb{Q}}$ is equal to $(\mathcal{E}'_\alpha)_\alpha$ as inductive systems and that $\varinjlim_\alpha \Psi_{\alpha, \mathbb{Q}} = \Psi'$.

Suppose first that we have shown the equality $(\mathcal{E}_\alpha)_{\alpha, \mathbb{Q}} = (\mathcal{E}'_\alpha)_\alpha$. Then, for any $\alpha \in \mathbb{Z}_{(p)}^r$, there exists some $\beta \in \mathbb{Z}_{(p)}^r$ such that both $\Psi_{\alpha, \mathbb{Q}}, \Psi'$ define morphism of the form $F^* \mathcal{E}_{\alpha, \mathbb{Q}} \rightarrow \mathcal{E}_{\beta, \mathbb{Q}}$. Then we have $\Psi_{\alpha, \mathbb{Q}}|_{\mathcal{X}} = \Psi'|_{\mathcal{X}}$ because they are equal when Z is empty (which follows from the definition of (4.2)), and this equality implies the equality $\Psi_{\alpha, \mathbb{Q}} = \Psi'$ as morphisms $F^* \mathcal{E}_{\alpha, \mathbb{Q}} \rightarrow \mathcal{E}_{\beta, \mathbb{Q}}$. Hence we have $\varinjlim_\alpha \Psi_{\alpha, \mathbb{Q}} = \Psi'$. So, to prove the claims in the previous paragraph, it suffices to prove the equality $(\mathcal{E}_\alpha)_{\alpha, \mathbb{Q}} = (\mathcal{E}'_\alpha)_\alpha$ as inductive systems.

Take a chart $(\overline{X}_0, \{t_i\}_{i=1}^r)$ for (\overline{X}, Z) in the sense of Section 2.3 which is smooth over $k[\mu_n]$ and let $(\overline{X}_\bullet, M_{\overline{X}_\bullet}), (\overline{X}_{\bullet\bullet}, M_{\overline{X}_{\bullet\bullet}})$ be the simplicial semi-resolution, the bisimplicial resolution of $(\overline{X}, Z)^{1/n}$, respectively. Let $(\overline{\mathcal{X}}_{\circ, \bullet}, M_{\overline{\mathcal{X}}_{\circ, \bullet}}), (\overline{\mathcal{X}}_{\circ, \bullet\bullet}, M_{\overline{\mathcal{X}}_{\circ, \bullet\bullet}})$ be the log étale lifts of $(\overline{X}_\bullet, M_{\overline{X}_\bullet}), (\overline{X}_{\bullet\bullet}, M_{\overline{X}_{\bullet\bullet}})$ over $(\overline{\mathcal{X}}_\circ, \mathcal{Z}_\circ)$ and let us put $(\overline{\mathcal{X}}_\bullet, M_{\overline{\mathcal{X}}_\bullet}) := (\overline{\mathcal{X}}_{\circ, \bullet}, M_{\overline{\mathcal{X}}_{\circ, \bullet}}) \otimes_{W(k)} O_K, (\overline{\mathcal{X}}_{\bullet\bullet}, M_{\overline{\mathcal{X}}_{\bullet\bullet}}) := (\overline{\mathcal{X}}_{\circ, \bullet\bullet}, M_{\overline{\mathcal{X}}_{\circ, \bullet\bullet}}) \otimes_{W(k)} O_K$. Then $M_{\overline{\mathcal{X}}_{\bullet\bullet}}$ is associated to some $(2, 2)$ -truncated bisimplicial relative simple normal crossing divisor $\mathcal{Z}_{\bullet\bullet} = \bigcup_{i=1}^r \mathcal{Z}_{\bullet\bullet, i}$ compatible with transition maps. (Here $\mathcal{Z}_{\bullet\bullet, i}$ is characterized as the smooth subdivisor of $\mathcal{Z}_{\bullet\bullet}$ which is homeomorphic to the inverse image of \mathcal{Z}_i .) Let us put $\mathcal{X}_\bullet := \mathcal{X} \times_{\overline{\mathcal{X}}} \overline{\mathcal{X}}_\bullet, \mathcal{X}_{\bullet\bullet} := \mathcal{X} \times_{\overline{\mathcal{X}}} \overline{\mathcal{X}}_{\bullet\bullet}$. Let us denote by $\text{NM}_{\overline{\mathcal{X}}_K}^\dagger$ (resp. $\text{NM}_{\overline{\mathcal{X}}_{\bullet, K}}^\dagger, \text{NM}_{\overline{\mathcal{X}}_{\bullet\bullet, K}}^\dagger$) the category of locally free $j^\dagger \mathcal{O}_{\overline{\mathcal{X}}_K}$ -modules (resp. $j^\dagger \mathcal{O}_{\overline{\mathcal{X}}_{\bullet, K}}$ -modules, $j^\dagger \mathcal{O}_{\overline{\mathcal{X}}_{\bullet\bullet, K}}$ -modules) of finite rank endowed with integrable connections, where j denotes the morphism $\mathcal{X}_K \hookrightarrow \overline{\mathcal{X}}_K$ (resp. $\mathcal{X}_{\bullet, K} \hookrightarrow \overline{\mathcal{X}}_{\bullet, K}, \mathcal{X}_{\bullet\bullet, K} \hookrightarrow \overline{\mathcal{X}}_{\bullet\bullet, K}$). Let us denote the restriction of $(\mathcal{E}_\alpha)_\alpha$ to $\overline{\mathcal{X}}_\bullet$ by $(\mathcal{E}_{\alpha, \bullet})_\alpha$ and the restriction of $(\mathcal{E}'_\alpha, \nabla'_\alpha)_\alpha$ to $\overline{\mathcal{X}}_{\bullet, K}$ by $(\mathcal{E}'_{\alpha, \bullet}, \nabla'_{\alpha, \bullet})_\alpha$. By rigid analytic faithfully flat descent, it suffices to prove the following: Via the equivalence between the \mathbb{Q} -linearization of the category of coherent sheaves on $\overline{\mathcal{X}}_\bullet$ and the category of coherent sheaves on $\overline{\mathcal{X}}_{\bullet, K}$, we have the equality $(\mathcal{E}_{\alpha, \bullet})_{\alpha, \mathbb{Q}} = (\mathcal{E}'_{\alpha, \bullet})_\alpha$.

Suppose that $((\mathcal{E}_\alpha)_\alpha, \Psi)$ is sent to $(\mathcal{E}'', \Psi'')_\mathbb{Q} \in F\text{-Latt}((\overline{\mathcal{X}}, \mathcal{Z})^{1/n})_\mathbb{Q}^\circ$ by the first functor of (4.32). Let us consider the diagram

$$(4.33) \quad F\text{-Latt}((\overline{\mathcal{X}}, \mathcal{Z})^{1/n})_\mathbb{Q}^\circ \xrightarrow{\quad} F\text{-Isoc}((\overline{\mathcal{X}}, Z)^{1/n})^\circ \longrightarrow \text{Par-}F\text{-Isoc}^{\log}(\overline{\mathcal{X}}, Z)_{\mathbf{0}\text{-ss}} \\ \xrightarrow{\text{induced by (3.23)}} \text{Isoc}^\dagger(X, \overline{X}) \xrightarrow{\quad} \text{NM}_{\mathcal{X}_K}^\dagger \longrightarrow \text{NM}_{\mathcal{X}_{\bullet, K}}^\dagger,$$

where the first two functors are induced by the second and the third ones in (4.32). Then, by definition, (\mathcal{E}'', Ψ'') is sent by (4.33) to $j^\dagger(\mathcal{E}'_{\alpha, \bullet}, \nabla'_{\alpha, \bullet})$. By definition of the functors in (4.33), it is rewritten as

$$(4.34) \quad F\text{-Latt}((\overline{\mathcal{X}}, \mathcal{Z})^{1/n})_\mathbb{Q}^\circ \\ \xrightarrow{\quad} F\text{-Latt}(\overline{\mathcal{X}}_\bullet \times_{\overline{\mathcal{X}}} (\overline{\mathcal{X}}, \mathcal{Z})^{1/n})^\circ \xrightarrow{\quad} F\text{-Latt}(\overline{\mathcal{X}}_{\bullet\bullet})_\mathbb{Q}^\circ \xrightarrow{(4.2) \text{ for } \overline{\mathcal{X}}_{\bullet\bullet}} F\text{-Isoc}(\overline{\mathcal{X}}_{\bullet\bullet})^\circ \\ \xrightarrow{\quad} \text{Isoc}(\overline{\mathcal{X}}_{\bullet\bullet}) \longrightarrow \text{Isoc}^\dagger(X_{\bullet\bullet}, \overline{X}_{\bullet\bullet}) \xrightarrow{(2.51)^{-1}} \text{Isoc}^\dagger(X_\bullet, \overline{X}_\bullet) \xrightarrow{\quad} \text{NM}_{\mathcal{X}_{\bullet, K}}^\dagger.$$

Furthermore, by the definition of the inverse of (2.51) given in (2.52), the composite $\text{Isoc}(\overline{X}_{\bullet\bullet}) \longrightarrow \text{NM}_{\overline{\mathcal{X}}_{\bullet,K}}^\dagger$ in (4.34) is rewritten as

$$\text{Isoc}(\overline{X}_{\bullet\bullet}) \xrightarrow{\subset} \text{NM}_{\overline{\mathcal{X}}_{\bullet\bullet,K}} \xrightarrow{j^\dagger} \text{NM}_{\overline{\mathcal{X}}_{\bullet\bullet,K}}^\dagger \xrightarrow{g} \text{NM}_{\overline{\mathcal{X}}_{\bullet,K}}^\dagger,$$

where the last functor g is defined as follows: An object $E_{\bullet\bullet}$ in $\text{NM}_{\overline{\mathcal{X}}_{\bullet\bullet,K}}^\dagger$, regarded as an object $E_{\bullet 0}$ in $\text{NM}_{\overline{\mathcal{X}}_{\bullet 0,K}}^\dagger$ endowed with an equivariant $\mu_n^{r(\bullet+1)}$ -action, is sent to $(\pi_* E_{\bullet 0})^{\mu_n^{r(\bullet+1)}}$, where π is the morphism $\overline{\mathcal{X}}_{\bullet 0} \longrightarrow \overline{\mathcal{X}}_\bullet$. Then, for any $\alpha \in \mathbb{Z}_{(p)}^r$, this is rewritten as

$$(4.35) \quad \begin{aligned} \text{Isoc}(\overline{X}_{\bullet\bullet}) &\xrightarrow{\subset} \text{NM}_{\overline{\mathcal{X}}_{\bullet\bullet,K}} \xrightarrow{f_\alpha} \text{LNM}_{\overline{\mathcal{X}}_{\bullet\bullet,K}} \\ &\xrightarrow{g'} \text{LNM}_{\overline{\mathcal{X}}_{\bullet,K}} \xrightarrow{j^\dagger} \text{NM}_{\overline{\mathcal{X}}_{\bullet,K}}^\dagger, \end{aligned}$$

where f_α is the functor $- \otimes_{\mathcal{O}_{\overline{\mathcal{X}}_{\bullet\bullet}}} \mathcal{O}_{\overline{\mathcal{X}}_{\bullet\bullet}}(\sum_{i=1}^r [n\alpha_i] \mathcal{Z}_{\bullet\bullet,i})$ (endowed with canonical extension or restriction of the connection) and g' is defined as follows, as in the case of g : An object $E_{\bullet\bullet}$ in $\text{LNM}_{\overline{\mathcal{X}}_{\bullet\bullet,K}}$, regarded as an object $E_{\bullet 0}$ in $\text{LNM}_{\overline{\mathcal{X}}_{\bullet 0,K}}$ endowed with an equivariant $\mu_n^{r(\bullet+1)}$ -action, is sent to $(\pi_* E_{\bullet 0})^{\mu_n^{r(\bullet+1)}}$.

Now let us assume that the object $(\mathcal{E}'', \Psi'')_{\mathbb{Q}} \in F\text{-Latt}((\overline{\mathcal{X}}, \mathcal{Z})^{1/n})_{\mathbb{Q}}^\circ$ is sent by the first two functors in (4.34) as $(\mathcal{E}'', \Psi'')_{\mathbb{Q}} \mapsto (\mathcal{E}_\bullet'', \Psi_\bullet'')_{\mathbb{Q}} \mapsto (\mathcal{E}_{\bullet\bullet}'', \Psi_{\bullet\bullet}'')_{\mathbb{Q}}$. First, by definition of \mathcal{E}'' , we have $\mathcal{E}_\alpha = \mathbf{a}(\mathcal{E}'')_\alpha$ and the functor \mathbf{a} is compatible with etale localization. So we have

$$(4.36) \quad \mathcal{E}_{\alpha,\bullet} = \mathbf{a}(\mathcal{E}_\bullet'')_\alpha = \pi_*(\mathcal{E}_{\bullet 0}'' \otimes_{\mathcal{O}_{\overline{\mathcal{X}}_{\bullet 0}}} (\sum_i [n\alpha_i] \mathcal{Z}_{\bullet 0,i}))^{\mu_n^{r(\bullet+1)}}.$$

The image of $(\mathcal{E}_{\bullet\bullet}'', \Psi_{\bullet\bullet}'')_{\mathbb{Q}}$ by the functor

$$F\text{-Latt}(\overline{\mathcal{X}}_{\bullet\bullet}) \longrightarrow F\text{-Isoc}(\overline{\mathcal{X}}_{\bullet\bullet}) \longrightarrow \text{Isoc}(\overline{X}_{\bullet\bullet}) \longrightarrow \text{NM}_{\overline{\mathcal{X}}_{\bullet\bullet,K}}$$

(where the first two functors are as in (4.34) and the last functor is as in (4.35)) has the form $(\mathcal{E}_{\bullet\bullet,\mathbb{Q}}'', \nabla_{\bullet\bullet})$, by definition of Crew's functor (4.2) which is given in [9]. If we apply the functor $g' \circ f_\alpha$, we obtain an object in $\text{LNM}_{\overline{\mathcal{X}}_{\bullet,K}}$ of the form $((\mathcal{E}_{\alpha,\bullet})_{\mathbb{Q}}, \nabla_{\alpha,\bullet})$ by definition of the functors f_α, g' and (4.36). Hence we have

$$(4.37) \quad j^\dagger((\mathcal{E}_{\alpha,\bullet})_{\mathbb{Q}}, \nabla_{\alpha,\bullet}) = j^\dagger(\mathcal{E}'_{\alpha,\bullet}, \nabla'_{\alpha,\bullet}).$$

By (4.37), we see that $j^\dagger((\mathcal{E}_{\alpha,\bullet})_{\mathbb{Q}}, \nabla_{\alpha,\bullet})$ comes from an overconvergent isocrystal. So $((\mathcal{E}_{\alpha,\bullet})_{\mathbb{Q}}, \nabla_{\alpha,\bullet})$ defines an object in $\text{Isoc}^{\log}(\overline{\mathcal{X}}_\bullet \times_{\overline{\mathcal{X}}} (\overline{\mathcal{X}}, Z))$. Since there exists the canonical morphism of functors $f_\alpha \longrightarrow f_\beta$ for $\alpha = (\alpha_i)_i, \beta = (\beta_i)_i$ with $\alpha_i \leq \beta_i (\forall i)$, we see that $((\mathcal{E}_{\alpha,\bullet})_{\mathbb{Q}}, \nabla_{\alpha,\bullet})$ for $\alpha \in \mathbb{Z}_{(p)}^r$ form an inductive system, and it is easy to see from the definition of f_α 's that it is a parabolic log convergent isocrystal. Moreover, Since $(\mathcal{E}_{\bullet\bullet,\mathbb{Q}}'', \nabla_{\bullet\bullet})$ has exponents in $\mathbf{0}$ with semisimple residues,

$f_\alpha(\mathcal{E}_{\bullet,\bullet,\mathbb{Q}}'', \nabla_{\bullet,\bullet})$ has exponents in $\prod_{i=1}^r \{-[n\alpha_i]\}$ with semisimple residues, and then we see that $((\mathcal{E}_{\alpha,\bullet})_{\mathbb{Q}}, \nabla_{\alpha,\bullet})$ has exponents in $\prod_{i=1}^r \{-\frac{[n\alpha_i]}{n} + \frac{j}{n} \mid 0 \leq j \leq n-1\} \subseteq \prod_{i=1}^r ([-\alpha_i, -\alpha_i+1] \cap \mathbb{Z}_{(p)})$ with semisimple residues. Hence $((\mathcal{E}_{\alpha,\bullet})_{\mathbb{Q}}, \nabla_{\alpha,\bullet})_\alpha$ forms an object in $\text{Par-Isoc}^{\log}(\overline{X}_\bullet \times_{\overline{X}} (\overline{X}, Z))_{0\text{-ss}}$. Since so is $(\mathcal{E}'_{\alpha,\bullet}, \nabla'_{\alpha,\bullet})$ and we have the isomorphism (4.37), they are isomorphic by the equivalence (3.10). So we have the desired isomorphism $(\mathcal{E}_{\alpha,\bullet})_{\alpha,\mathbb{Q}} = (\mathcal{E}'_{\alpha,\bullet})_\alpha$.

5 Unit-rootness and generic semistability

In this section, we prove several propositions which give interpretations of unit-rootness in terms of certain semistability called generic semistability, and prove the equivalences (0.14)–(0.23). The proof uses the results in [18], [8] and [9] and the constructions up to the previous section.

First let us give a review on some results proven in [18], [8] and [9]. (We also give a slight generalization of them which we need in this paper.) In this section, let π be a fixed uniformizer of O_K . For a perfect field l containing k , we put $O(l) := W(l) \otimes_{W(k)} O_K$, $K(l) := \text{Frac } O(l)$. We denote the endomorphism $F_\circ \otimes \sigma$ (where F_\circ is the endomorphism on $W(k)$ lifting the q -th power map on l) on $O(l)$ by F , and denote the induced endomorphism on $K(l)$ by the same letter. An F -isocrystal on l is defined to be a pair (E, Ψ) consisting of a finite dimensional $K(l)$ -vector space E and a F -linear endomorphism Ψ . Then, by Dieudonné-Manin classification theorem, any F -isocrystal (E, Ψ) has the decomposition $(E, \Psi) := \bigoplus_{\lambda \in \mathbb{Q}} (E_\lambda, \Psi_\lambda)$ as F -isocrystals such that, for each $\lambda = a/b$, $E \otimes_{K(l)} K(l^{\text{alg}})$ has a basis consisting of elements e with $\Psi^b(e) = \pi^a$. The Newton polygon of (E, Ψ) is the convex polygon defined as the convex closure of the points $(\sum_{\lambda \leq \nu} \dim E_\lambda, \sum_{\lambda \leq \nu} \lambda \dim E_\lambda)$ ($\nu \in \mathbb{Q}$) in the plane \mathbb{R}^2 . (So it has the endpoint $(\dim E, \sum_\lambda \lambda \dim E_\lambda)$). For an F -isocrystal $(E, \Psi) \neq 0$, we put $\mu(E) := \frac{\sum_\lambda \lambda \dim E_\lambda}{\dim E}$. We say that the Newton polygon of (E, Ψ) has pure slope λ when it is the straight line connecting $(0, 0)$ and $(\dim E, \lambda \dim E)$.

For a fine log scheme (X, M_X) separated of finite type over k , an object $(\mathcal{E}, \Psi) \in F\text{-Isoc}(X, M_X)$ and a perfect valued point $x = \text{Spec } l \rightarrow X$, we can pull back (\mathcal{E}, Ψ) by x to an object of the category of convergent F -isocrystal on $(x, M_X|_x)$ over $(\text{Spf } O(l), W(M_X|_x)|_{\text{Spec } O(l)})$ (where $W(M_X|_x)$ denotes the canonical lift of the log structure $M_X|_x$ on $x = \text{Spec } l$ into $\text{Spf } W(l)$), which is canonically equivalent to the category of F -isocrystals on l . We denote this object by $x^*(\mathcal{E}, \Psi)$ or $(x^*\mathcal{E}, x^*\Psi)$. We call the Newton polygon of $x^*(\mathcal{E}, \Psi)$ the Newton polygon of (\mathcal{E}, Ψ) at x and we put $\mu_x(\mathcal{E}) := \mu(x^*\mathcal{E})$.

Next, let X be a smooth scheme separated of finite type over k and assume that it is liftable to a p -adic formal scheme \mathcal{X}_\circ separated smooth of finite type over $\text{Spf } W(k)$ which is endowed with a lift $F_\circ : \mathcal{X}_\circ \rightarrow \mathcal{X}_\circ$ of the q -th power Frobenius endomorphism on X compatible with $(\sigma|_{W(k)})^* : \text{Spf } W(k) \rightarrow \text{Spf } W(k)$. Let us put $\mathcal{X} := \mathcal{X}_\circ \otimes O_K$, $F := F_\circ \otimes \sigma^* : \mathcal{X} \rightarrow \mathcal{X}$. Then the category $F\text{-Latt}(\mathcal{X})$ of

F -lattices on \mathcal{X} was defined in the beginning of the previous section. Also, we define the notion of F -vector bundle on \mathcal{X}_K as a pair (\mathcal{E}, Ψ) of a locally free $\mathcal{O}_{\mathcal{X}_K}$ -module \mathcal{E} endowed with an isomorphism $\Psi : F^*\mathcal{E} \xrightarrow{\cong} \mathcal{E}$. (It is called a rigid F -bundle in [51].) We denote the category of F -vector bundles on \mathcal{X}_K by $F\text{-Vect}(\mathcal{X}_K)$. Note that, via the equivalence $\text{Coh}(\mathcal{X})_{\mathbb{Q}} \xrightarrow{\cong} \text{Coh}(\mathcal{X}_K)$, we have the canonical inclusion $F\text{-Latt}(\mathcal{X})_{\mathbb{Q}} \hookrightarrow F\text{-Vect}(\mathcal{X}_K)$. (We do not know whether they are equal or not.) When we are given an object $(\mathcal{E}, \Psi) \in F\text{-Vect}(\mathcal{X}_K)$ and a perfect valued point $x = \text{Spec } l \rightarrow X$, we have the canonical lift $O(x) := \text{Spf } O(l) \rightarrow \mathcal{X}$ and so we have the map $\text{Spm } K(x) \rightarrow \mathcal{X}_K$. If we pull back (\mathcal{E}, Ψ) by this map, we obtain an F -isocrystal on l , which we denote by $x^*(\mathcal{E}, \Psi)$ or $(x^*\mathcal{E}, x^*\Psi)$. Also in this case, we call the Newton polygon of $x^*(\mathcal{E}, \Psi)$ the Newton polygon of (\mathcal{E}, Ψ) at x and we put $\mu_x(\mathcal{E}) := \mu(x^*\mathcal{E})$.

Assume moreover that there exists a fine log structure M_X on X , $M_{\mathcal{X}}$ on \mathcal{X} on \mathcal{X} with $M_{\mathcal{X}}|_X = M_X$ and that F induces the endomorphism $F : (\mathcal{X}, M_{\mathcal{X}}) \rightarrow (\mathcal{X}, M_{\mathcal{X}})$ lifting the q -th power Frobenius on (X, M_X) . Then, by definition, we have the commutative diagram

$$\begin{array}{ccc} F\text{-Isoc}(X, M_X) & \longrightarrow & F\text{-Vect}(\mathcal{X}_K) \\ x^* \downarrow & & x^* \downarrow \\ (F\text{-isocrystals on } l) & \longequal{\quad} & (F\text{-isocrystals on } l), \end{array}$$

where the top horizontal arrow is the functor of realization at $(\mathcal{X}, M_{\mathcal{X}})$. It is easy to see that, for a fixed object (\mathcal{E}, Ψ) in $F\text{-Isoc}(X)$ or $F\text{-Vect}(\mathcal{X}_K)$, the Newton polygon of (\mathcal{E}, Ψ) at x and $\mu_x(\mathcal{E})$ for a perfect field valued point $x \rightarrow X$ depends only on the image of x in X . Hence we can speak of the Newton polygon of (\mathcal{E}, Ψ) at x and the value $\mu_x(\mathcal{E})$ at any point x of X .

Now let us recall several results on F -lattices which are due to Grothendieck, Katz and Crew. The first one is Grothendieck's specialization theorem ([18, 2.3], [8, 1.6]):

Proposition 5.1. *let X be a smooth scheme separated of finite type over k and assume that it is liftable to a p -adic formal scheme \mathcal{X}_{\circ} separated of finite type over $\text{Spf } W(k)$ which is endowed with a lift $F_{\circ} : \mathcal{X}_{\circ} \rightarrow \mathcal{X}_{\circ}$ of the q -th power Frobenius endomorphism on X compatible with $(\sigma|_{W(k)})^* : \text{Spf } W(k) \rightarrow \text{Spf } W(k)$. Let us put $\mathcal{X} := \mathcal{X}_{\circ} \otimes O_K$, $F := F_{\circ} \otimes \sigma^* : \mathcal{X} \rightarrow \mathcal{X}$, let (\mathcal{E}, Ψ) be an object in $F\text{-Latt}(\mathcal{X})_{\mathbb{Q}}$ and let P be a convex polygon. Then the set of points of X at which the Newton polygon of (\mathcal{E}, Ψ) lies on or above P is Zariski closed.*

The next one is the constance of $\mu_x(\mathcal{E})$ [8, 1.7]:

Proposition 5.2. *Let X, \mathcal{X}, F be as above and suppose that X is connected. Then, for a non-zero object (\mathcal{E}, Ψ) in $F\text{-Latt}(\mathcal{X})_{\mathbb{Q}}$, $\mu_x(\mathcal{E})$ is the same for all points x of X . (In this case, we put $\mu(\mathcal{E}) := \mu_x(\mathcal{E})$.)*

The next one, which is a weaker form of [18, 2.6.2], is a ‘generic’ Newton filtration theorem:

Proposition 5.3. *Let X, \mathcal{X}, F be as above and let (\mathcal{E}, Ψ) be an object in $F\text{-Latt}(\mathcal{X})_{\mathbb{Q}}$ such that its Newton polygon at $x \in X$ is not a straight line (has a break point) and independent of $x \in X$. Then, on a dense open formal subscheme \mathcal{U} of \mathcal{X} , (\mathcal{E}, Ψ) admits a non-trivial saturated subobject (\mathcal{E}', Ψ') with $\mu(\mathcal{E}') < \mu(\mathcal{E})$. (Where a subobject (\mathcal{E}', Ψ') of (\mathcal{E}, Ψ) is called saturated if the quotient $(\mathcal{E}/\mathcal{E}', \overline{\Psi})$ ($\overline{\Psi}$ is the morphism induced by Ψ) is again an object in $F\text{-Latt}(\mathcal{X})_{\mathbb{Q}}$.) When $\mathcal{E}_{\mathbb{Q}}$ is endowed with an integrable connection for which Ψ is horizontal, $\nabla|_{\mathcal{E}'_{\mathbb{Q}}}$ defines an integrable connection on $\mathcal{E}'_{\mathbb{Q}}$ for which Ψ' is horizontal.*

The next proposition, which is shown in [9, 2.5.1] and [8, 2.3], fills a possible gap in the inclusion $F\text{-Latt}(\mathcal{X})_{\mathbb{Q}} \hookrightarrow F\text{-Vect}(\mathcal{X}_K)$ generically:

Proposition 5.4. *Let X, \mathcal{X}, F be as above and assume that \mathcal{X} is affine. Then, for any object (\mathcal{E}, Ψ) in $F\text{-Vect}(\mathcal{X}_K)$, there exists a dense open formal subscheme \mathcal{U} of \mathcal{X} such that $(\mathcal{E}, \Psi)|_{\mathcal{U}} \in F\text{-Vect}(\mathcal{X}_K)$ is contained in $F\text{-Latt}(\mathcal{U})_{\mathbb{Q}}$. Moreover, in the case where $\dim X = 1$, we can take $\mathcal{U} = \mathcal{X}$.*

With more strong condition on the Newton polygons, we have the following result, which is proved in [9, 2.5.1–2.6]:

Proposition 5.5. *Let X, \mathcal{X}, F be as above and assume that \mathcal{X} is affine. Let (\mathcal{E}, Ψ) be an object in $F\text{-Vect}(\mathcal{X}_K)$ such that, for any point $x \in X$, the Newton polygon of it at x has pure slope 0. Then there exists a unit-root F -lattice (\mathcal{E}_0, Ψ_0) on \mathcal{X} with $(\mathcal{E}_0, \Psi_0)_{\mathbb{Q}} = (\mathcal{E}, \Psi)$.*

Using the above results, we see the following proposition, which is a kind of specialization theorem for log convergent F -isocrystals.

Proposition 5.6. *Let $X \hookrightarrow \overline{X}$ be an open immersion of connected smooth k -varieties such that $Z = \overline{X} \setminus X$ is a simple normal crossing divisor, and let (\mathcal{E}, Ψ) be an object in $F\text{-Isoc}^{\log}(\overline{X}, Z)$. Let η be the generic point of \overline{X} . Then, for any $x \in \overline{X}$, the Newton polygon of (\mathcal{E}, Ψ) at x lies on or above the Newton polygon of (\mathcal{E}, Ψ) at η and we have the equality $\mu_x(\mathcal{E}) = \mu_{\eta}(\mathcal{E})$. (In particular, $\mu_x(\mathcal{E})$ is independent of x . So we put $\mu(\mathcal{E}) := \mu_x(\mathcal{E})$ in this situation.)*

Before the proof, we introduce one terminology. For a fine log scheme (T, M_T) separated of finite type over k , a strong lift of (T, M_T) is a fine log formal scheme $(\mathcal{T}, M_{\mathcal{T}})$ separated of finite type over $\text{Spf } O_K$ endowed with an endomorphism $F : (\mathcal{T}, M_{\mathcal{T}}) \rightarrow (\mathcal{T}, M_{\mathcal{T}})$ such that there exists a lift $(\mathcal{T}_o, M_{\mathcal{T}_o})$ of (T, M_T) over $\text{Spf } W(k)$ endowed with a $(\sigma|_{W(k)})^*$ -linear endomorphism $F_o : (\mathcal{T}_o, M_{\mathcal{T}_o}) \rightarrow (\mathcal{T}_o, M_{\mathcal{T}_o})$ lifting the q -th power map on (T, M_T) satisfying $(\mathcal{T}, M_{\mathcal{T}}) = (\mathcal{T}_o, M_{\mathcal{T}_o}) \otimes_{W(k)} O_K, F = F_o \otimes_{(\sigma|_{W(k)})^*} \sigma^*$. When (T, M_T) is log smooth over k , $(\mathcal{T}, M_{\mathcal{T}})$ is called a strong smooth lift of (T, M_T) if we can take $(\mathcal{T}_o, M_{\mathcal{T}_o})$ to be log smooth over $\text{Spf } W(k)$.

Proof. Let $Z = \bigcup_{i=1}^r Z_i$ be the decomposition of Z into irreducible components. Since we may work locally around x , we may suppose that each Z_i is defined as the zero locus of some element t_i in \overline{X} and that $x \in \bigcap_{i=1}^r Z_i =: Y_r$. (So $r = r(x)$ is the number of irreducible components which contains x .)

First we reduce the proof of the proposition to the case $r = r(x) = 0$ by descending induction on r . Let us denote the log structure on \overline{X} associated to Z by $M_{\overline{X}}$ and let us put $M_{Y_r} := M_{\overline{X}}|_{Y_r}$. Then M_{Y_r} is equal to the log structure associated to zero map $\mathbb{N}^r \rightarrow \mathcal{O}_{Y_r}$. Let V_x be a non-empty smooth open subscheme of the closure of x in Y_r . Then, after shrinking V_x properly, it admits a strong smooth lift \mathcal{V}_x , and if we define $M_{\mathcal{V}_x}$ to be the log structure associated to the zero map $\mathbb{N}^r \rightarrow \mathcal{O}_{\mathcal{V}_x}$, $(\mathcal{V}_x, M_{\mathcal{V}_x})$ is a strong lift (no more smooth) of $(V_x, M_{Y_r}|_{V_x})$. Then (\mathcal{E}, Ψ) naturally induces an object $(\mathcal{E}_{\mathcal{V}_x}, \Psi_{\mathcal{V}_x}) \in F\text{-Vect}(\mathcal{V}_{x,K})$, and by shrinking \mathcal{V}_x , we may assume that $(\mathcal{E}_{\mathcal{V}_x}, \Psi_{\mathcal{V}_x})$ belongs to $F\text{-Latt}(\mathcal{V}_x)_{\mathbb{Q}}$, by Proposition 5.4. Then, by Propositions 5.1 and 5.2, we have the following after shrinking V_x further: For any $y \in V_x$, $\mu_y(\mathcal{E}) = \mu_x(\mathcal{E})$ and that the Newton polygon of \mathcal{E} at y is the same as the Newton polygon of \mathcal{E} at x . So, by replacing x by a closed point of V_x , we may assume that x is a closed point to prove the proposition. Next, let us put $Y_{r-1} := \bigcap_{i=1}^{r-1} Z_i$, $M_{Y_{r-1}} := M_{\overline{X}}|_{Y_{r-1}}$ and let $M'_{Y_{r-1}}$ be the log structure on Y_{r-1} associated to Y_r . Then $M_{Y_{r-1}}$ is equal to the direct sum (in the category of fine log structures) of $M'_{Y_{r-1}}$ and the log structure associated to zero map $\mathbb{N}^{r-1} \rightarrow \mathcal{O}_{Y_{r-1}}$. Let us take an affine smooth curve C on Y_{r-1} passing through x which is transversal to Y_r . Then $(C, M'_{Y_{r-1}}|_C)$, being log smooth, admits a strong smooth lift $(\mathcal{C}, M'_{\mathcal{C}})$ when we shrink C appropriately, keeping the condition $x \in C$. Then, if we define $M_{\mathcal{C}}$ to be the direct sum of $M'_{\mathcal{C}}$ and the log structure associated to the zero map $\mathbb{N}^{r-1} \rightarrow \mathcal{O}_{\mathcal{C}}$, $(\mathcal{C}, M_{\mathcal{C}})$ is a strong lift (no more smooth) of $(C, M_{Y_{r-1}}|_C)$. Then (\mathcal{E}, Ψ) naturally induces an object $(\mathcal{E}_{\mathcal{C}}, \Psi_{\mathcal{C}}) \in F\text{-Vect}(\mathcal{C}_K)$ and by Proposition 5.4, it belongs to $F\text{-Latt}(\mathcal{C})_{\mathbb{Q}}$. Let y be the generic point of C . Then, by Propositions 5.1 and 5.2, we have $\mu_y(\mathcal{E}) = \mu_x(\mathcal{E})$ and that the Newton polygon of \mathcal{E} at x lies on or above the Newton polygon of \mathcal{E} at y . Hence the proposition for x is reduced to the proposition for y . Since $y \in Y_{r-1} \setminus Y_r$, we have $r(y) = r(x) - 1$. So, by descending induction, we can reduce the proof of the theorem to the case $r(x) = 0$.

In the case $r(x) = 0$, the theorem is essentially due to Crew [8, 2.1]. Here we give a proof for the convenience of the reader, since the statement here and that in [8, 2.1] are slightly different. By the first argument in the previous paragraph, we may assume that x is a closed point of X . Since we may work locally around x , we may assume that $X = \overline{X}$ and that this admits a strong smooth lift \mathcal{X} . Then (\mathcal{E}, Ψ) naturally induces an object $(\mathcal{E}_{\mathcal{X}}, \Psi_{\mathcal{X}}) \in F\text{-Vect}(\mathcal{X}_K)$. By Proposition 5.4, we have a dense open formal subscheme $\mathcal{U} \subseteq \mathcal{X}$ such that $(\mathcal{E}_{\mathcal{X}}, \Psi_{\mathcal{X}})|_{\mathcal{U}}$ belongs to $F\text{-Latt}(\mathcal{U})_{\mathbb{Q}}$. Then, by Propositions 5.1 and 5.2, we have the following after shrinking \mathcal{U} further: For any $y \in \mathcal{U} := \mathcal{U} \otimes k$, $\mu_y(\mathcal{E}) = \mu_{\eta}(\mathcal{E})$ and that the Newton polygon of \mathcal{E} at y lies on or above the Newton polygon of \mathcal{E} at η . Next let us take an affine smooth curve C on X passing through x with $U \cap C \neq \emptyset$ such that C admits a strong smooth lift \mathcal{C} . Then (\mathcal{E}, Ψ) naturally induces an object $(\mathcal{E}_{\mathcal{C}}, \Psi_{\mathcal{C}}) \in F\text{-Vect}(\mathcal{C}_K)$ and by Proposition

5.4, it belongs to $F\text{-Latt}(\mathcal{C})_{\mathbb{Q}}$. Let y be the generic point of C , which belongs to $U \cap C$. Then, by Propositions 5.1 and 5.2, we have $\mu_y(\mathcal{E}) = \mu_x(\mathcal{E})$ and that the Newton polygon of \mathcal{E} at x is on or above the Newton polygon of \mathcal{E} at y . Hence we have shown that $\mu_x(\mathcal{E}) = \mu_\eta(\mathcal{E})$ and that the Newton polygon of \mathcal{E} at x lies on or above the Newton polygon of \mathcal{E} at η , as desired. \square

Corollary 5.7. *Let (\overline{X}, Z) , η be as in Proposition 5.6 and let (\mathcal{E}, Ψ) be an object in $F\text{-Isoc}^{\log}(\overline{X}, Z)$ whose Newton polygon at η has pure slope s . Then, for any $x \in \overline{X}$, the Newton polygon of (\mathcal{E}, Ψ) at x has pure slope s .*

The following is the generic Newton filtration theorem for (log) convergent F -isocrystals.

Proposition 5.8. *Let (\overline{X}, Z) , X , η be as in Proposition 5.6 and let (\mathcal{E}, Ψ) be an object in $F\text{-Isoc}^{\log}(\overline{X}, Z)$ such that its Newton polygon at η is not a straight line (has a break point) and independent of $x \in X$. Then, on an dense open subscheme of X , (\mathcal{E}, Ψ) admits a non-trivial subobject (\mathcal{E}', Ψ') with $\mu(\mathcal{E}') < \mu(\mathcal{E})$.*

Proof. Since we may shrink \overline{X} , we may assume that $X = \overline{X}$ and that it admits a strong smooth lift \mathcal{X} . Then (\mathcal{E}, Ψ) naturally induces an object $(\mathcal{E}_{\mathcal{X}}, \Psi_{\mathcal{X}})$ in $F\text{-Vect}(\mathcal{X}_K)$ endowed with an integrable connection ∇ for which Ψ is horizontal. Then we may assume that $(\mathcal{E}_{\mathcal{X}}, \Psi_{\mathcal{X}})$ belongs to $F\text{-Latt}(\mathcal{X})_{\mathbb{Q}}$ by Proposition 5.4 and by Proposition 5.3, $(\mathcal{E}_{\mathcal{X}}, \Psi_{\mathcal{X}})$ admits a non-trivial subobject $(\mathcal{E}'_{\mathcal{X}}, \Psi'_{\mathcal{X}})$ with $\mu(\mathcal{E}') < \mu(\mathcal{E})$ such that $\nabla|_{\mathcal{E}'_{\mathcal{X}}}$ defines an integrable connection on $\mathcal{E}'_{\mathcal{X}}$ for which $\Psi'_{\mathcal{X}}$ is horizontal. Then the triple $(\mathcal{E}'_{\mathcal{X}}, \nabla|_{\mathcal{E}'_{\mathcal{X}}}, \Psi'_{\mathcal{X}})$ naturally induces a non-trivial subobject (\mathcal{E}', Ψ') of (\mathcal{E}, Ψ) with $\mu(\mathcal{E}') < \mu(\mathcal{E})$. (The ‘convergence’ of $\nabla|_{\mathcal{E}'_{\mathcal{X}}}$ follows from that of ∇ .) \square

Now we give the definition of generic semistability for log convergent F -isocrystals.

Definition 5.9. *Let $X \hookrightarrow \overline{X}$ be an open immersion of connected smooth k -varieties such that $Z = \overline{X} \setminus X$ is a simple normal crossing divisor, and let (\mathcal{E}, Ψ) be an object in $F\text{-Isoc}^{\log}(\overline{X}, Z)$. Then (\mathcal{E}, Ψ) is called generically semistable (gss) if, for any dense open subscheme $U \subseteq X$, $(\mathcal{E}, \Psi)|_U \in F\text{-Isoc}(U)$ does not admit any non-trivial subobject (\mathcal{E}', Ψ') with $\mu(\mathcal{E}') < \mu(\mathcal{E})$. We denote the full subcategory of $F\text{-Isoc}^{\log}(\overline{X}, Z)$ consisting of generically semistable objects (\mathcal{E}, Ψ) with $\mu(\mathcal{E}) = 0$ by $F\text{-Isoc}^{\log}(\overline{X}, Z)^{\text{gss}, \mu=0}$. In the case where $\overline{X} = \coprod_i \overline{X}_i$ is not necessarily connected, we define the category $F\text{-Isoc}^{\log}(\overline{X}, Z)^{\text{gss}, \mu=0}$ as the product of the categories $F\text{-Isoc}^{\log}(\overline{X}_i, \overline{X}_i \cap Z)^{\text{gss}, \mu=0}$.*

Then we have the following:

Proposition 5.10. *Let $X \hookrightarrow \overline{X}$ be an open immersion of connected smooth k -varieties such that $Z = \overline{X} \setminus X$ is a simple normal crossing divisor. Then we have the canonical equivalence*

$$(5.1) \quad F\text{-Isoc}(\overline{X})^{\circ} \xrightarrow{=} F\text{-Isoc}^{\log}(\overline{X}, Z)^{\text{gss}, \mu=0}$$

Proof. Let η be the generic point of \overline{X} . We see by definition that, for any object (\mathcal{E}, Ψ) in $F\text{-Isoc}(\overline{X})^\circ$, the Newton polygon of it at η has pure slope 0. Hence so is for any dense open $U \subseteq X$ and any subobject of $(\mathcal{E}, \Psi)|_U$ in $F\text{-Isoc}(U)$. Hence (\mathcal{E}, Ψ) is generically semistable with $\mu(\mathcal{E}) = 0$, that is, the functor (5.1) is well-defined as the canonical inclusion.

It suffices to show the essential surjectivity of the functor (5.1) to prove the proposition. So let us take an object (\mathcal{E}, Ψ) in $F\text{-Isoc}^{\log}(\overline{X}, Z)$ generically semistable with $\mu(\mathcal{E}) = 0$. Then, by Propositions 5.1, 5.2 and 5.4, there exists an open dense subscheme $U \subseteq X$ such that the Newton polygon of (\mathcal{E}, Ψ) at x is independent of x . So, if the Newton polygon of (\mathcal{E}, Ψ) at η is not a straight line, (\mathcal{E}, Ψ) is not generically semistable by Proposition 5.8. So the Newton polygon of (\mathcal{E}, Ψ) at η has pure slope $\mu(\mathcal{E}) = 0$. Then, by Corollary 5.7, the Newton polygon of (\mathcal{E}, Ψ) at x has pure slope 0 at any point x in \overline{X} . Therefore, to see that (\mathcal{E}, Ψ) is in $F\text{-Isoc}(\overline{X})^\circ$, it suffices to show that it is contained in $F\text{-Isoc}(\overline{X})$. To see this, we may work locally. So we can assume that X, \overline{X} are affine and that there exists a smooth lift $\overline{\mathcal{X}}$ of \overline{X} , a lift $\mathcal{Z} = \bigcup_{i=1}^r \mathcal{Z}_i$ of Z which is a relative simple normal crossing divisor of $\overline{\mathcal{X}}$ and a lift $F : (\overline{\mathcal{X}}, \mathcal{Z}) \rightarrow (\overline{\mathcal{X}}, \mathcal{Z})$ of the q -th power Frobenius endomorphism on (\overline{X}, Z) . Moreover, we may assume that each \mathcal{Z}_i is defined as the zero locus of some element t_i in $\Gamma(\overline{\mathcal{X}}, \mathcal{O}_{\overline{\mathcal{X}}})$. Then (\mathcal{E}, Ψ) naturally induces an object $(\mathcal{E}_{\overline{\mathcal{X}}}, \Psi_{\overline{\mathcal{X}}})$ in $F\text{-Vect}(\overline{\mathcal{X}}_K)$ endowed with an integrable log connection $\nabla : \mathcal{E}_{\overline{\mathcal{X}}} \rightarrow \mathcal{E}_{\overline{\mathcal{X}}} \otimes \Omega_{\overline{\mathcal{X}}_K}^1(\log \mathcal{Z}_K)$. Then, to see that (\mathcal{E}, Ψ) is contained in $F\text{-Isoc}(\overline{X})$, it suffices to see that the image of ∇ is contained in $\mathcal{E}_{\overline{\mathcal{X}}} \otimes \Omega_{\overline{\mathcal{X}}_K}^1$. By Proposition 5.5, there exists a unit-root F -lattice (\mathcal{E}_0, Ψ_0) with $(\mathcal{E}_0, \Psi_0)_{\mathbb{Q}} := (\mathcal{E}_{0, \mathbb{Q}}, \Psi_{0, \mathbb{Q}}) = (\mathcal{E}_{\overline{\mathcal{X}}}, \Psi_{\overline{\mathcal{X}}})$. In the following, we identify $\mathcal{E}_{\overline{\mathcal{X}}}, \mathcal{E}_0, \Omega_{\overline{\mathcal{X}}_K}^1, \Omega_{\overline{\mathcal{X}}_K}^1(\log \mathcal{Z}_K)$ with the set of global sections of them. For $n \in \mathbb{Z}$, let us put

$$\Omega_n(\mathcal{E}_0) := \left(\sum_{i=1}^r p^n \mathcal{E}_0 d \log t_i \right) + \mathcal{E}_{\overline{\mathcal{X}}} \otimes \Omega_{\overline{\mathcal{X}}_K}^1 \subseteq \mathcal{E}_{\overline{\mathcal{X}}} \otimes \Omega_{\overline{\mathcal{X}}_K}^1(d \log \mathcal{Z}_K).$$

We prove the inclusion $\nabla(\mathcal{E}_0) \subseteq \Omega_n(\mathcal{E}_0)$ ($n \in \mathbb{Z}$) by induction on n . First, since \mathcal{E}_0 is finitely generated as $\mathcal{O}_{\overline{\mathcal{X}}}$ -module and we have $\bigcup_{n \in \mathbb{Z}} \Omega_n(\mathcal{E}_0) = \mathcal{E}_{\overline{\mathcal{X}}} \otimes \Omega_{\overline{\mathcal{X}}_K}^1(d \log \mathcal{Z}_K)$, we have $\nabla(\mathcal{E}_0) \subseteq \Omega_n$ for sufficiently small n . Next, assume that we have $\nabla(\mathcal{E}_0) \subseteq \Omega_n(\mathcal{E}_0)$. Since we have

$$F^* \Omega_n(\mathcal{E}_0) = \left(\sum_{i=1}^r p^{n+1} F^* \mathcal{E}_0 d \log t_i \right) + F^* \mathcal{E}_{\overline{\mathcal{X}}} \otimes \Omega_{\overline{\mathcal{X}}_K}^1 =: \Omega_{n+1}(F^* \mathcal{E}_0),$$

we have $F^* \nabla(F^* \mathcal{E}_0) \subseteq \Omega_{n+1}(F^* \mathcal{E}_0)$, where we denoted the pull-back by F of $\nabla : \mathcal{E}_{\overline{\mathcal{X}}} \rightarrow \mathcal{E}_{\overline{\mathcal{X}}} \otimes \Omega_{\overline{\mathcal{X}}_K}^1(\log \mathcal{Z}_K)$ by $F^* \nabla$. By sending this by Ψ , we obtain the inclusion $\nabla(\mathcal{E}_0) \subseteq \Omega_{n+1}(\mathcal{E}_0)$, as desired. Then we have

$$\nabla(\mathcal{E}_0) \subseteq \bigcap_{n \in \mathbb{Z}} \Omega_n(\mathcal{E}_0) = \mathcal{E}_{\overline{\mathcal{X}}} \otimes \Omega_{\overline{\mathcal{X}}_K}^1,$$

and so we obtain the desired inclusion $\nabla(\mathcal{E}_{\overline{\mathcal{X}}}) \subseteq \mathcal{E}_{\overline{\mathcal{X}}} \otimes \Omega_{\overline{\mathcal{X}}_K}^1$. So we are done. \square

Remark 5.11. Since the category $F\text{-Isoc}(\overline{X})$ is a full subcategory of $F\text{-Isoc}^{\log}(\overline{X}, Z)$, the equivalence (5.1) induces the equivalence

$$F\text{-Isoc}(\overline{X})^{\circ} \xrightarrow{=} F\text{-Isoc}(\overline{X})^{\text{gss}, \mu=0}.$$

So we have the interpretation of unit-rootness in terms of generic semistability.

Next we define the notion of generic semistability for certain stacky categories and give a stacky version of Proposition 5.10.

Definition 5.12. Let $X \hookrightarrow \overline{X}$ be an open immersion of connected smooth k -varieties with $\overline{X} \setminus X =: Z = \bigcup_{i=1}^r Z_i$ a simple normal crossing divisor (each Z_i being irreducible).

- (1) Let \mathcal{G}_X be the category of finite etale Galois covering of X . Let $\varphi_Y : Y \rightarrow X$ be an object in \mathcal{G}_X , let $G_Y := \text{Aut}(Y/X)$ and let \overline{Y}^{sm} be the smooth locus of the normalization \overline{Y} in \overline{X} in $k(Y)$. (Then we have the quotient stack $[\overline{Y}^{\text{sm}}/G_Y]$ and the canonical log structure $M_{[\overline{Y}^{\text{sm}}/G_Y]}$ which are defined in Section 2.1.) Then an object (\mathcal{E}, Ψ) in $F\text{-Isoc}^{\log}([\overline{Y}^{\text{sm}}/G_Y], M_{[\overline{Y}^{\text{sm}}/G_Y]})$ is called generically semistable (gss) if, for any dense open subscheme $U \subseteq X$, the image $(\mathcal{E}|_U, \Psi|_U)$ of (\mathcal{E}, Ψ) by the restriction functor

$$F\text{-Isoc}^{\log}([\overline{Y}^{\text{sm}}/G_Y], M_{[\overline{Y}^{\text{sm}}/G_Y]}) \rightarrow F\text{-Isoc}^{\log}([\varphi_Y^{-1}(U)/G_Y], M_{[\overline{Y}^{\text{sm}}/G_Y]|_{[\varphi_Y^{-1}(U)/G_Y]}}) = F\text{-Isoc}(U)$$

does not admit any non-trivial subobject (\mathcal{E}', Ψ') with $\mu(\mathcal{E}') < \mu(\mathcal{E}|_U)$. (Note that, in the definition above, the quantity $\mu(\mathcal{E}|_U)$ is independent of the choice of U . So we denote it simply by $\mu(\mathcal{E})$ in the sequel.) We denote the full subcategory of $F\text{-Isoc}^{\log}([\overline{Y}^{\text{sm}}/G_Y], M_{[\overline{Y}^{\text{sm}}/G_Y]})$ consisting of generically semistable objects (\mathcal{E}, Ψ) with $\mu(\mathcal{E}) = 0$ by

$$F\text{-Isoc}^{\log}([\overline{Y}^{\text{sm}}/G_Y], M_{[\overline{Y}^{\text{sm}}/G_Y]})^{\text{gss}, \mu=0}.$$

- (2) For $n \in \mathbb{N}$ with $(n, p) = 1$, let $(\overline{X}, Z)^{1/n}$ be the stack of n -th roots of (\overline{X}, Z) . (Then we have the canonical log structure $M_{(\overline{X}, Z)^{1/n}}$, which is defined in Section 2.3.) Then an object (\mathcal{E}, Ψ) in $F\text{-Isoc}^{\log}((\overline{X}, Z)^{1/n}, M_{(\overline{X}, Z)^{1/n}})$ is called generically semistable (gss) if, for any dense open subscheme $U \subseteq X$, the image $(\mathcal{E}|_U, \Psi|_U)$ of (\mathcal{E}, Ψ) by the restriction functor

$$F\text{-Isoc}^{\log}((\overline{X}, Z)^{1/n}, M_{(\overline{X}, Z)^{1/n}}) \rightarrow F\text{-Isoc}^{\log}(U \times_{\overline{X}} (\overline{X}, Z)^{1/n}, M_{(\overline{X}, Z)^{1/n}}|_{U \times_{\overline{X}} (\overline{X}, Z)^{1/n}}) = F\text{-Isoc}(U)$$

does not admit any non-trivial subobject (\mathcal{E}', Ψ') with $\mu(\mathcal{E}') < \mu(\mathcal{E}|_U)$. (Note that, in the definition above, the quantity $\mu(\mathcal{E}|_U)$ is independent of the choice of U . So we denote it simply by $\mu(\mathcal{E})$ in the sequel.) We denote the full subcategory of $F\text{-Isoc}^{\log}((\overline{X}, Z)^{1/n}, M_{(\overline{X}, Z)^{1/n}})$ consisting of generically semistable objects (\mathcal{E}, Ψ) with $\mu(\mathcal{E}) = 0$ by $F\text{-Isoc}^{\log}((\overline{X}, Z)^{1/n}, M_{(\overline{X}, Z)^{1/n}})^{\text{gss}, \mu=0}$.

Proposition 5.13. *Let the notations be as in Definition 5.12. Then we have the canonical equivalences*

$$(5.2) \quad \varinjlim_{Y \rightarrow X \in \mathcal{G}_X} F\text{-Isoc}([\overline{Y}^{\text{sm}}/G_Y])^\circ \xrightarrow{=} \varinjlim_{Y \rightarrow X \in \mathcal{G}_X} F\text{-Isoc}^{\log}([\overline{Y}^{\text{sm}}/G_Y], M_{[\overline{Y}^{\text{sm}}/G_Y]})^{\text{gss}, \mu=0},$$

$$(5.3) \quad \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} F\text{-Isoc}([\overline{Y}^{\text{sm}}/G_Y])^\circ \xrightarrow{=} \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} F\text{-Isoc}^{\log}([\overline{Y}^{\text{sm}}/G_Y], M_{[\overline{Y}^{\text{sm}}/G_Y]})^{\text{gss}, \mu=0}$$

(where \mathcal{G}_X^t denotes the category of finite etale Galois tame covering of X (tamely ramified along the discrete valuations v_i ($1 \leq i \leq r$) corresponding to the generic point of Z_i)) and

$$(5.4) \quad \varinjlim_{(n,p)=1} F\text{-Isoc}((\overline{X}, Z)^{1/n})^\circ \xrightarrow{=} \varinjlim_{(n,p)=1} F\text{-Isoc}^{\log}((\overline{X}, Z)^{1/n}, M_{(\overline{X}, Z)^{1/n}})^{\text{gss}, \mu=0}.$$

Proof. For $m = 0, 1, 2$, let $\overline{Y}_m^{\text{sm}}$ be the $(m+1)$ -fold fiber product of \overline{Y}^{sm} over $[\overline{Y}^{\text{sm}}/G_Y]$ and denote the resulting 2-truncated simplicial scheme by $\overline{Y}_\bullet^{\text{sm}}$. Then we have the equivalences

$$\begin{aligned} & \varinjlim_{Y \rightarrow X \in \mathcal{G}_X} F\text{-Isoc}([\overline{Y}^{\text{sm}}/G_Y])^\circ \xrightarrow{=} \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} F\text{-Isoc}(\overline{Y}_\bullet^{\text{sm}})^\circ \\ & \xrightarrow{(5.1)} \varinjlim_{Y \rightarrow X \in \mathcal{G}_X} F\text{-Isoc}^{\log}(\overline{Y}_\bullet^{\text{sm}}, M_{\overline{Y}_\bullet^{\text{sm}}})^{\text{gss}, \mu=0} \\ & \xrightarrow{=} \varinjlim_{Y \rightarrow X \in \mathcal{G}_X} F\text{-Isoc}^{\log}([\overline{Y}^{\text{sm}}/G_Y], M_{[\overline{Y}^{\text{sm}}/G_Y]})^{\text{gss}, \mu=0}, \end{aligned}$$

where $F\text{-Isoc}^{\log}(\overline{Y}_\bullet^{\text{sm}}, M_{\overline{Y}_\bullet^{\text{sm}}})^{\text{gss}, \mu=0}$ denotes the full subcategory of $F\text{-Isoc}^{\log}(\overline{Y}_\bullet^{\text{sm}}, M_{\overline{Y}_\bullet^{\text{sm}}})$ consisting of objects whose restriction to $F\text{-Isoc}^{\log}(\overline{Y}_m^{\text{sm}}, M_{\overline{Y}_m^{\text{sm}}})$ are contained in $F\text{-Isoc}^{\log}(\overline{Y}_m^{\text{sm}}, M_{\overline{Y}_m^{\text{sm}}})^{\text{gss}, \mu=0}$ for $m = 0, 1, 2$. So we have shown (5.2), and we can prove (5.3) exactly in the same way. Next, take a chart $(\overline{X}_0, \{t_i\}_{1 \leq i \leq r})$ of (\overline{X}, Z) in the sense of Section 2.3 and for $n \in \mathbb{N}$ with $(n, p) = 1$, let $(\overline{X}_{\bullet\bullet}^{(n)}, M_{\overline{X}_{\bullet\bullet}^{(n)}})$ be the bisimplicial resolution of $(\overline{X}, Z)^{1/n}$ associated to the chart $(\overline{X}_0, \{t_i\}_{1 \leq i \leq r})$. Then we have the equivalences

$$\begin{aligned} & \varinjlim_{(n,p)=1} F\text{-Isoc}((\overline{X}, Z)^{1/n})^\circ \xrightarrow{=} \varinjlim_{(n,p)=1} F\text{-Isoc}(\overline{X}_{\bullet\bullet}^{(n)})^\circ \\ & \xrightarrow{(5.1)} \varinjlim_{(n,p)=1} F\text{-Isoc}^{\log}(\overline{X}_{\bullet\bullet}^{(n)}, M_{\overline{X}_{\bullet\bullet}^{(n)}})^{\text{gss}, \mu=0} \\ & \xrightarrow{=} \varinjlim_{(n,p)=1} F\text{-Isoc}^{\log}((\overline{X}, Z)^{1/n}, M_{(\overline{X}, Z)^{1/n}})^{\text{gss}, \mu=0}. \end{aligned}$$

So we are done. \square

Next we define the notion of generic semistability for adjusted parabolic log convergent F -isocrystals and give a parabolic version of Proposition 5.10.

Definition 5.14. *Let $X \hookrightarrow \overline{X}$ be an open immersion of connected smooth k -varieties with $\overline{X} \setminus X =: Z = \bigcup_{i=1}^r Z_i$ a simple normal crossing divisor (each Z_i being irreducible). Then an object $((\mathcal{E}_\alpha)_\alpha, \Psi)$ in $\text{Par-}F\text{-Isoc}^{\log}(\overline{X}, Z)_0$ is called generically semistable (gss) if, for any dense open subscheme $U \subseteq X$, the image $(\mathcal{E}|_U, \Psi|_U)$ of $((\mathcal{E}_\alpha)_\alpha, \Psi)$ by the restriction functor*

$$\text{Par-}F\text{-Isoc}^{\log}(\overline{X}, Z)_0 \longrightarrow \text{Par-}F\text{-Isoc}^{\log}(U, \emptyset)_0 = F\text{-Isoc}(U)$$

does not admit any non-trivial subobject (\mathcal{E}', Ψ') with $\mu(\mathcal{E}') < \mu(\mathcal{E}|_U)$. (Note that, in the definition above, the quantity $\mu(\mathcal{E}|_U)$ is independent of the choice of U . So we denote it by $\mu((\mathcal{E}_\alpha)_\alpha)$ in the sequel.) In the following, we denote the full subcategory of $\text{Par-}F\text{-Isoc}^{\log}(\overline{X}, Z)_0$ consisting of generically semistable objects $((\mathcal{E}_\alpha)_\alpha, \Psi)$ with $\mu((\mathcal{E}_\alpha)_\alpha) = 0$ by $\text{Par-}F\text{-Isoc}^{\log}(\overline{X}, Z)_0^{\text{gss}, \mu=0}$.

Theorem 5.15. *Let the notations be as in Definition 5.14. Then we have the canonical equivalence of categories*

$$(5.5) \quad \text{Par-}F\text{-Isoc}^{\log}(\overline{X}, Z)_0^{\circ\text{-ss}} \xrightarrow{=} \text{Par-}F\text{-Isoc}^{\log}(\overline{X}, Z)_0^{\text{gss}, \mu=0}.$$

Proof. This is an immediate consequence of the equivalence (5.3) and the equivalences (3.14), (3.16). \square

As an immediate consequence of Propositions 5.13, 5.15, we have the following, which is a p -adic analogues of (0.2) and (0.3) and (0.5) which includes the notion of ‘stability’:

Corollary 5.16. *Let the notations be as in Definition 5.12. Then we have the equivalences*

$$\begin{aligned} \text{Rep}_{K^\sigma}^{\text{fin}}(\pi_1(X)) &\xrightarrow{=} \varinjlim_{Y \rightarrow X \in \mathcal{G}_X} F\text{-Isoc}^{\log}([\overline{Y}^{\text{sm}}/G_Y], M_{[\overline{Y}^{\text{sm}}/G_Y]})^{\text{gss}, \mu=0}, \\ \text{Rep}_{K^\sigma}(\pi_1^t(X)) &\xrightarrow{=} \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} F\text{-Isoc}^{\log}([\overline{Y}^{\text{sm}}/G_Y], M_{[\overline{Y}^{\text{sm}}/G_Y]})^{\text{gss}, \mu=0}, \\ \text{Rep}_{K^\sigma}(\pi_1^t(X)) &\xrightarrow{=} \varinjlim_{(n,p)=1} F\text{-Isoc}^{\log}((\overline{X}, Z)^{1/n}, M_{(\overline{X}, Z)^{1/n}})^{\text{gss}, \mu=0}, \\ \text{Rep}_{K^\sigma}(\pi_1^t(X)) &\xrightarrow{=} \text{Par-}F\text{-Isoc}^{\log}(\overline{X}, Z)^{\text{gss}, \mu=0}. \end{aligned}$$

Next we prove ‘ F -lattice versions’ of Propositions 5.10, 5.13 and 5.15. Let X be a connected smooth scheme separated of finite type over k and assume that it is liftable to a p -adic formal scheme \mathcal{X}_\circ separated of finite type over $\text{Spf } W(k)$ which is endowed with a lift $F_\circ : \mathcal{X}_\circ \rightarrow \mathcal{X}_\circ$ of the q -th power Frobenius endomorphism on X compatible with $(\sigma|_{W(k)})^* : \text{Spf } W(k) \rightarrow \text{Spf } W(k)$. Let us put $\mathcal{X} := \mathcal{X}_\circ \otimes_{O_K}$,

$F := F_\circ \otimes \sigma^* : \mathcal{X} \longrightarrow \mathcal{X}$. Then an object (\mathcal{E}, Ψ) in $F\text{-Latt}(\mathcal{X})_\mathbb{Q}$ is said to be generically semistable (gss) if, for any open dense formal subscheme $\mathcal{U}_\circ \longrightarrow \mathcal{X}_\circ$, $(\mathcal{E}|_\mathcal{U}, \Psi|_\mathcal{U})$ (where $\mathcal{U} := \mathcal{U}_\circ \otimes_{W(k)} O_K$) admits no non-trivial saturated subobject (\mathcal{E}', Ψ') with $\mu(\mathcal{E}') < \mu(\mathcal{E})$. We denote the full subcategory of $F\text{-Latt}(\mathcal{X})_\mathbb{Q}$ consisting of generically semistable objects (\mathcal{E}, Ψ) with $\mu(\mathcal{E}) = 0$ by $(F\text{-Latt}(\mathcal{X})_\mathbb{Q})^{\text{gss}, \mu=0}$. When X is not necessarily connected, $(F\text{-Latt}(\mathcal{X})_\mathbb{Q})^{\text{gss}, \mu=0}$ denotes the full subcategory of $F\text{-Latt}(\mathcal{X})_\mathbb{Q}$ which are generically semistable with $\mu = 0$ on each connected component of \mathcal{X} . Then the F -lattice version of Proposition 5.10 is the following proposition (which is an immediate consequence of the results of Crew and Katz quoted in the beginning of this section):

Proposition 5.17. *Let the notations be as above. Then we have the equivalence*

$$(5.6) \quad F\text{-Latt}(\mathcal{X})_\mathbb{Q}^\circ = (F\text{-Latt}(\mathcal{X})_\mathbb{Q})^{\text{gss}, \mu=0}$$

Proof. It is easy to see that the left hand side is contained in the right hand side, and we may assume that \mathcal{X} is connected. Let η be the generic point of X and let us take an object (\mathcal{E}, Ψ) in $(F\text{-Latt}(\mathcal{X})_\mathbb{Q})^{\text{gss}, \mu=0}$. Then, by Propositions 5.1 and 5.2, there exists an dense open $\mathcal{U}_\circ \longrightarrow \mathcal{X}_\circ$ such that the Newton polygon of $(\mathcal{E}|_\mathcal{U}, \Psi|_\mathcal{U})$ (where $\mathcal{U} := \mathcal{U}_\circ \otimes_{W(k)} O_K$) at any point is equal to that at η . Then, by Proposition 5.3, the generic semistability of (\mathcal{E}, Ψ) implies that the Newton polygon of (\mathcal{E}, Ψ) at η has pure slope $\mu(\mathcal{E}) = 0$. Again by Propositions 5.1 and 5.2, we see that the Newton polygon of (\mathcal{E}, Ψ) at any point has pure slope 0. Then, by Proposition 5.5, we see that (\mathcal{E}, Ψ) is contained in $F\text{-Latt}(\mathcal{X})_\mathbb{Q}^\circ$. So we are done. \square

Next we define the notion of generic semistability for F -lattices on stacks and give an F -lattice version of Theorem 5.13.

Let the notations be as in Section 4, the paragraphs after (4.3), before Theorem 4.1. Then, for an object $Y \longrightarrow X$ in \mathcal{G}_X^t , the ind fine log algebraic stack $([\overline{\mathcal{Y}}^{\text{sm}}/G_Y], M_{[\overline{\mathcal{Y}}^{\text{sm}}/G_Y]}) = \varinjlim_a ([\overline{\mathcal{Y}}_a^{\text{sm}}/G_Y], M_{[\overline{\mathcal{Y}}_a^{\text{sm}}/G_Y]})$, the endomorphism F (lift of Frobenius) on it and the category $F\text{-Latt}([\overline{\mathcal{Y}}^{\text{sm}}/G_Y])$ of F -lattices on $[\overline{\mathcal{Y}}^{\text{sm}}/G_Y]$ are defined there. The ind algebraic stack $(\overline{\mathcal{X}}, \mathcal{Z})^{1/n} = \varinjlim_a (\overline{\mathcal{X}}, \mathcal{Z})_a^{1/n}$ endowed with the endomorphism F (lift of Frobenius) and the category $F\text{-Latt}((\overline{\mathcal{X}}, \mathcal{Z})^{1/n})$ of F -lattices on $(\overline{\mathcal{X}}, \mathcal{Z})^{1/n}$ are also defined there.

An object (\mathcal{E}, Ψ) in $F\text{-Latt}([\overline{\mathcal{Y}}^{\text{sm}}/G_Y])_\mathbb{Q}$ (resp. $F\text{-Latt}((\overline{\mathcal{X}}, \mathcal{Z})^{1/n})_\mathbb{Q}$) is called generically semistable (gss) if, for any dense open formal subscheme $\mathcal{U}_\circ \subseteq \mathcal{X}_\circ$, the image of (\mathcal{E}, Ψ) by the restriction functor

$$\begin{aligned} F\text{-Latt}([\overline{\mathcal{Y}}^{\text{sm}}/G_Y])_\mathbb{Q} &\longrightarrow F\text{-Latt}(\mathcal{U} \times_{\mathcal{X}} [\overline{\mathcal{Y}}^{\text{sm}}/G_Y])_\mathbb{Q} \xrightarrow{=} F\text{-Latt}(\mathcal{U})_\mathbb{Q} \\ (\text{resp. } F\text{-Latt}((\overline{\mathcal{X}}, \mathcal{Z})^{1/n})_\mathbb{Q} &\longrightarrow F\text{-Latt}(\mathcal{U} \times_{\mathcal{X}} (\overline{\mathcal{X}}, \mathcal{Z})^{1/n})_\mathbb{Q} \xrightarrow{=} F\text{-Latt}(\mathcal{U})_\mathbb{Q}) \end{aligned}$$

admits no non-trivial saturated subobject (\mathcal{E}', Ψ') with $\mu(\mathcal{E}') < \mu(\mathcal{E}|_\mathcal{U})$. (Note that $\mu(\mathcal{E}|_\mathcal{U})$ does not depend on \mathcal{E} . Hence we denote it by $\mu(\mathcal{E})$.) We denote the full subcategory of $F\text{-Latt}([\overline{\mathcal{Y}}^{\text{sm}}/G_Y])_\mathbb{Q}$ (resp. $F\text{-Latt}((\overline{\mathcal{X}}, \mathcal{Z})^{1/n})_\mathbb{Q}$) consisting of generically semistable objects (\mathcal{E}, Ψ) with $\mu(\mathcal{E}) = 0$ by $F\text{-Latt}([\overline{\mathcal{Y}}^{\text{sm}}/G_Y])_\mathbb{Q}^{\text{gss}, \mu=0}$ (resp.

$F\text{-Latt}((\overline{\mathcal{X}}, \mathcal{Z})^{1/n})_{\mathbb{Q}}^{\text{gss}, \mu=0}$). Then we have the following proposition, which is the F -lattice version of Proposition 5.13:

Theorem 5.18. *Let the notations be as above. Then we have the canonical equivalences*

$$(5.7) \quad \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} F\text{-Latt}([\overline{\mathcal{Y}}^{\text{sm}}/G_Y]_{\mathbb{Q}})^{\circ} \xrightarrow{=} \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} F\text{-Latt}([\overline{\mathcal{Y}}^{\text{sm}}/G_Y]_{\mathbb{Q}})^{\text{gss}, \mu=0},$$

$$(5.8) \quad \varinjlim_{(n,p)=1} F\text{-Latt}((\overline{\mathcal{X}}, \mathcal{Z})^{1/n})_{\mathbb{Q}}^{\circ} \xrightarrow{=} \varinjlim_{(n,p)=1} F\text{-Latt}((\overline{\mathcal{X}}, \mathcal{Z})^{1/n})_{\mathbb{Q}}^{\text{gss}, \mu=0}.$$

Proof. Let $\overline{\mathcal{Y}}_{\bullet}^{\text{sm}}$ be as in the proof of Theorem 4.1. Then we have the functors

$$(5.9) \quad \begin{aligned} \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} F\text{-Latt}([\overline{\mathcal{Y}}^{\text{sm}}/G_Y]_{\mathbb{Q}})^{\circ} &\xrightarrow{=} \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} F\text{-Latt}(\overline{\mathcal{Y}}_{\bullet}^{\text{sm}})_{\mathbb{Q}}^{\circ} \\ &\longrightarrow \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} F\text{-Latt}(\overline{\mathcal{Y}}_{\bullet}^{\text{sm}})_{\mathbb{Q}}^{\text{gss}, \mu=0} \xrightarrow{=} \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} F\text{-Latt}([\overline{\mathcal{Y}}^{\text{sm}}/G_Y]_{\mathbb{Q}})^{\text{gss}, \mu=0}, \end{aligned}$$

where $F\text{-Latt}(\overline{\mathcal{Y}}_{\bullet}^{\text{sm}})_{\mathbb{Q}}^{\text{gss}, \mu=0}$ denotes the full subcategory of $F\text{-Latt}(\overline{\mathcal{Y}}_{\bullet}^{\text{sm}})_{\mathbb{Q}}$ consisting of objects whose restriction to $F\text{-Latt}(\overline{\mathcal{Y}}_m^{\text{sm}})_{\mathbb{Q}}$ are contained in $F\text{-Latt}(\overline{\mathcal{Y}}_m^{\text{sm}})_{\mathbb{Q}}^{\text{gss}, \mu=0}$ for $m = 0, 1, 2$. Let us prove that the second arrow in the above diagram is an equivalence. Let us denote the category of compatible system of objects in $F\text{-Latt}(\overline{\mathcal{Y}}_m^{\text{sm}})_{\mathbb{Q}}^{\circ}$ ($m = 0, 1, 2$) (resp. $F\text{-Latt}(\overline{\mathcal{Y}}_m^{\text{sm}})_{\mathbb{Q}}^{\text{gss}, \mu=0}$ ($m = 0, 1, 2$)) by $\{F\text{-Latt}(\overline{\mathcal{Y}}_m^{\text{sm}})_{\mathbb{Q}}^{\circ}\}_{m=0,1,2}$ (resp. $\{F\text{-Latt}(\overline{\mathcal{Y}}_m^{\text{sm}})_{\mathbb{Q}}^{\text{gss}, \mu=0}\}_{m=0,1,2}$.) Then we have the following commutative diagram

$$\begin{array}{ccc} \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} F\text{-Latt}(\overline{\mathcal{Y}}_{\bullet}^{\text{sm}})_{\mathbb{Q}}^{\circ} & \longrightarrow & \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} \{F\text{-Latt}(\overline{\mathcal{Y}}_m^{\text{sm}})_{\mathbb{Q}}^{\circ}\}_{m=0,1,2} \\ \downarrow & & \downarrow \\ \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} F\text{-Latt}(\overline{\mathcal{Y}}_{\bullet}^{\text{sm}})_{\mathbb{Q}}^{\text{gss}, \mu=0} & \longrightarrow & \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} \{F\text{-Latt}(\overline{\mathcal{Y}}_m^{\text{sm}})_{\mathbb{Q}}^{\text{gss}, \mu=0}\}_{m=0,1,2}, \end{array}$$

where the horizontal arrows are natural fully faithful inclusions, the left vertical arrow is the one in the diagram (5.9) and the right vertical arrow is defined as in the left vertical one. Then, Proposition 5.17 implies that the left vertical arrow is an equivalence. On the other hand, by the proof of Theorem 4.1, we see that the top horizontal arrow, which is equal to (4.14), is an equivalence. Hence the right vertical arrow is also an equivalence. So (5.9) is an equivalence, as desired.

Next, let $\overline{\mathcal{X}}_{\bullet\bullet}^{(n)}$ be also as in the proof of Theorem 4.1. Then we have the functors

$$\begin{aligned} \varinjlim_{(n,p)=1} F\text{-Latt}((\overline{\mathcal{X}}, \mathcal{Z})^{1/n})_{\mathbb{Q}}^{\circ} &\xrightarrow{=} \varinjlim_{(n,p)=1} F\text{-Latt}(\overline{\mathcal{X}}_{\bullet\bullet}^{(n)})_{\mathbb{Q}}^{\circ} \\ &\longrightarrow \varinjlim_{(n,p)=1} F\text{-Latt}(\overline{\mathcal{X}}_{\bullet\bullet}^{(n)})_{\mathbb{Q}}^{\text{gss}, \mu=0} \xrightarrow{=} \varinjlim_{(n,p)=1} F\text{-Latt}((\overline{\mathcal{X}}, \mathcal{Z})^{1/n})_{\mathbb{Q}}^{\text{gss}, \mu=0} \end{aligned}$$

(where $F\text{-Latt}(\overline{\mathcal{X}}_{\bullet\bullet}^{(n)})_{\mathbb{Q}}^{\text{gss}, \mu=0}$ denotes the full subcategory of $F\text{-Latt}(\overline{\mathcal{X}}_{\bullet\bullet}^{(n)})_{\mathbb{Q}}$ consisting of objects whose restriction to $F\text{-Latt}(\overline{\mathcal{X}}_{kl}^{(n)})_{\mathbb{Q}}$ are contained in $F\text{-Latt}(\overline{\mathcal{X}}_{kl}^{(n)})_{\mathbb{Q}}^{\text{gss}, \mu=0}$ for $k, l = 0, 1, 2$), and we can prove that the second arrow is an equivalence in the same way as in the previous paragraph, using the equivalence (4.22) instead of (4.14). So we are done. \square

Next we define the notion of generic semistability for locally abelian parabolic F -lattices and prove an F -lattice version of Proposition 5.15.

Let the notations be as in Definitions 4.3, 4.4. Then an object $((\mathcal{E}_\alpha)_\alpha, (\Psi_\alpha)_\alpha)$ in $\text{Par-}F\text{-Latt}(\overline{\mathcal{X}}, \mathcal{Z})_{\mathbb{Q}}$ is called generically semistable (gss) if, for any dense open formal subscheme $\mathcal{U}_\circ \subseteq \mathcal{X}_\circ$, the image $(\mathcal{E}|_{\mathcal{U}}, \Psi|_{\mathcal{U}})$ of $((\mathcal{E}_\alpha)_\alpha, (\Psi_\alpha)_\alpha)$ by the restriction functor

$$\text{Par-}F\text{-Latt}(\overline{\mathcal{X}}, \mathcal{Z})_{\mathbb{Q}} \longrightarrow \text{Par-}F\text{-Latt}(\mathcal{U} \times_{\overline{\mathcal{X}}} (\overline{\mathcal{X}}, \mathcal{Z}))_{\mathbb{Q}} \xrightarrow{=} F\text{-Latt}(\mathcal{U})_{\mathbb{Q}}$$

(where $\mathcal{U} := \mathcal{U}_\circ \otimes_{W(k)} O_K$) admits no non-trivial saturated subobject (\mathcal{E}', Ψ') with $\mu(\mathcal{E}') < \mu(\mathcal{E}|_{\mathcal{U}})$. (Note that $\mu(\mathcal{E}|_{\mathcal{U}})$ does not depend on \mathcal{E} . Hence we denote it by $\mu(\mathcal{E})$.) We denote the full subcategory of $\text{Par-}F\text{-Latt}(\overline{\mathcal{X}}, \mathcal{Z})_{\mathbb{Q}}$ consisting of generically semistable objects (\mathcal{E}, Ψ) with $\mu(\mathcal{E}) = 0$ by $\text{Par-}F\text{-Latt}(\overline{\mathcal{X}}, \mathcal{Z})_{\mathbb{Q}}^{\text{gss}, \mu=0}$. Then we have the following proposition, which is the F -lattice version of Proposition 5.15:

Proposition 5.19. *Let the notations be as above. Then we have the canonical equivalence of categories*

$$(5.10) \quad \text{Par-}F\text{-Latt}(\overline{\mathcal{X}}, \mathcal{Z})_{\mathbb{Q}}^{\circ} \xrightarrow{=} \text{Par-}F\text{-Latt}(\overline{\mathcal{X}}, \mathcal{Z})_{\mathbb{Q}}^{\text{gss}, \mu=0}.$$

Proof. Note that the equivalence 4.25

$$\varinjlim_{(n,p)=1} F\text{-Latt}((\overline{\mathcal{X}}, \mathcal{Z})^{1/n}) \xrightarrow{=} \text{Par-}F\text{-Latt}(\overline{\mathcal{X}}, \mathcal{Z})$$

preserves the generic semistabilities and the value of μ . Then the desired equivalence follows from the equivalence (5.8). \square

As an immediate consequence of Propositions 5.18, 5.19, we have the following, which is a p -adic analogues (F -lattice version) of (0.2) and (0.3) and (0.5) which includes the notion of stability:

Theorem 5.20. *Let the notations be as above. Then we have the equivalences*

$$\begin{aligned} \text{Rep}_{K^\sigma}(\pi_1^t(X)) &\xrightarrow{=} \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} F\text{-Latt}([\overline{\mathcal{Y}}^{\text{sm}}/G_Y])_{\mathbb{Q}}^{\text{gss}, \mu=0}. \\ \text{Rep}_{K^\sigma}(\pi_1^t(X)) &\xrightarrow{=} \varinjlim_{(n,p)=1} F\text{-Latt}((\overline{\mathcal{X}}, \mathcal{Z})^{1/n})_{\mathbb{Q}}^{\text{gss}, \mu=0}. \\ \text{Rep}_{K^\sigma}(\pi_1^t(X)) &\xrightarrow{=} \text{Par-}F\text{-Latt}(\overline{\mathcal{X}}, \mathcal{Z})_{\mathbb{Q}}^{\text{gss}, \mu=0}. \end{aligned}$$

Roughly speaking, Theorem 5.20 claims (when compared with Theorem 5.16) that we can forget the isocrystal structure (connection) in the categories on the right hand sides if we assume a strong liftability condition. But we have to put the lattice structure. Finally in this paper, we introduce the category of ‘ F -vector bundles on certain rigid analytic stacks’, ‘locally abelian parabolic F -vector bundles on certain log rigid analytic spaces’ and prove a p -adic analogues (F -vector bundle version) of (0.2) and (0.3) and (0.5) in which neither isocrystal structure and lattice structure appear, in the case of curves satisfying a strong liftability condition.

Let X be a connected smooth scheme separated of finite type over k and assume that it is liftable to a p -adic formal scheme \mathcal{X}_\circ separated of finite type over $\mathrm{Spf} W(k)$ which is endowed with a lift $F_\circ : \mathcal{X}_\circ \rightarrow \mathcal{X}_\circ$ of the q -th power Frobenius endomorphism on X compatible with $(\sigma|_{W(k)})^* : \mathrm{Spf} W(k) \rightarrow \mathrm{Spf} W(k)$. Let us put $\mathcal{X} := \mathcal{X}_\circ \otimes_{O_K} F := F_\circ \otimes \sigma^* : \mathcal{X} \rightarrow \mathcal{X}$. Then an object (\mathcal{E}, Ψ) in $F\text{-Vect}(\mathcal{X}_K)$ is said to be generically semistable (gss) if, for any open dense formal subscheme $\mathcal{U}_\circ \rightarrow \mathcal{X}_\circ$, $(\mathcal{E}|_{\mathcal{U}_K}, \Psi|_{\mathcal{U}_K})$ (where $\mathcal{U} := \mathcal{U}_\circ \otimes_{W(k)} O_K$) admits no non-trivial saturated subobject (\mathcal{E}', Ψ') with $\mu(\mathcal{E}') < \mu(\mathcal{E})$, where a subobject (\mathcal{E}', Ψ') of $(\mathcal{E}|_{\mathcal{U}_K}, \Psi|_{\mathcal{U}_K})$ is called saturated if the quotient $(\mathcal{E}|_{\mathcal{U}_K}/\mathcal{E}', \overline{\Psi|_{\mathcal{U}_K}})$ ($\overline{\Psi|_{\mathcal{U}_K}}$ is the morphism induced by $\Psi|_{\mathcal{U}_K}$) is again an object in $F\text{-Vect}(\mathcal{U}_K)$. (Note that $\mu(\mathcal{E}|_{\mathcal{U}_K})$ does not depend on \mathcal{E} . Hence we denote it by $\mu(\mathcal{E})$.) We denote the full subcategory of $F\text{-Vect}(\mathcal{X})_{\mathbb{Q}}$ consisting of generically semistable objects (\mathcal{E}, Ψ) with $\mu(\mathcal{E}) = 0$ by $F\text{-Vect}(\mathcal{X}_K)^{\text{gss}, \mu=0}$. When X is not necessarily connected, $F\text{-Vect}(\mathcal{X}_K)^{\text{gss}, \mu=0}$ denotes the full subcategory consisting of objects which are generically semistable with $\mu = 0$ on each connected component of X . Then we have the following proposition:

Proposition 5.21. *Let the notations be as above and assume $\dim X = 1$. Then we have the equivalence*

$$(5.11) \quad F\text{-Latt}(\mathcal{X})_{\mathbb{Q}}^{\circ} \xrightarrow{\cong} F\text{-Vect}(\mathcal{X}_K)^{\text{gss}, \mu=0}.$$

Proof. By Proposition 5.4, $F\text{-Vect}(\mathcal{X}_K)^{\text{gss}, \mu=0}$ is contained in $(F\text{-Latt}(\mathcal{X})_{\mathbb{Q}})^{\text{gss}, \mu=0}$ in the case of curves. Hence the proposition is reduced to Proposition 5.17. \square

Recall that, for a p -adic formal scheme \mathcal{S} separated of finite type over $\mathrm{Spf} O_K$, we have the canonical equivalence $\mathrm{Coh}(\mathcal{S})_{\mathbb{Q}} \simeq \mathrm{Coh}(\mathcal{S}_K)$ of the \mathbb{Q} -linearization of the category of coherent $\mathcal{O}_{\mathcal{S}}$ -modules and the category of coherent $\mathcal{O}_{\mathcal{S}_K}$ -modules. With this in mind, we give the following definitions: Let the notations be as in Section 4, the paragraphs after (4.3), before Definition 4.1. Then, for an object $Y \rightarrow X \in \mathcal{G}_Y^t$, the ind stack $[\overline{\mathcal{Y}}^{\mathrm{sm}}/G_Y] = \varinjlim_a [\overline{\mathcal{Y}}_a^{\mathrm{sm}}/G_Y]$ and the endomorphism F on it is defined. We define the category $F\text{-Vect}([\overline{\mathcal{Y}}^{\mathrm{sm}}/G_Y]_K)$ of ‘ F -vector bundles on $[\overline{\mathcal{Y}}^{\mathrm{sm}}/G_Y]_K$ ’ as the category of pairs (\mathcal{E}, Ψ) , where \mathcal{E} is an object in $\mathrm{Coh}([\overline{\mathcal{Y}}^{\mathrm{sm}}/G_Y]_{\mathbb{Q}})$ ($:=$ the \mathbb{Q} -linearization of the category of compatible families of coherent $\mathcal{O}_{[\overline{\mathcal{Y}}_a^{\mathrm{sm}}/G_Y]}$ -modules) such that, for any morphism $\mathcal{S} \rightarrow [\overline{\mathcal{Y}}^{\mathrm{sm}}/G_Y]$ from a p -adic formal scheme separated of finite type over $\mathrm{Spf} O_K$, $\mathcal{E}|_{\mathcal{S}} \in \mathrm{Coh}(\mathcal{O}_{\mathcal{S}})_{\mathbb{Q}} \simeq \mathrm{Coh}(\mathcal{O}_{\mathcal{S}_K})$ is a locally free $\mathcal{O}_{\mathcal{S}_K}$ -module of finite rank, and Ψ is an isomorphism $F^*\mathcal{E} \xrightarrow{\cong} \mathcal{E}$. (Attention: We

only defined the category of F -vector bundles on $[\overline{\mathcal{Y}}^{\text{sm}}/G_Y]_K$, not the rigid stack $[\overline{\mathcal{Y}}^{\text{sm}}/G_Y]_K$ itself.) In the same way, we define the category $F\text{-Vect}((\overline{\mathcal{X}}, \mathcal{Z})_K^{1/n})$ of ‘ F -vector bundles on $(\overline{\mathcal{X}}, \mathcal{Z})_K^{1/n}$ ’ as the category of pairs (\mathcal{E}, Ψ) , where \mathcal{E} is an object in $\text{Coh}((\overline{\mathcal{X}}, \mathcal{Z})_K^{1/n})_{\mathbb{Q}}$ ($:=$ the \mathbb{Q} -linearization of the category of compatible families of coherent $\mathcal{O}_{(\overline{\mathcal{X}}, \mathcal{Z})_K^{1/n}}$ -modules) such that, for any morphism $\mathcal{S} \rightarrow (\overline{\mathcal{X}}, \mathcal{Z})_K^{1/n}$ from a p -adic formal scheme separated of finite type over $\text{Spf } \mathcal{O}_K$, $\mathcal{E}|_{\mathcal{S}} \in \text{Coh}(\mathcal{O}_{\mathcal{S}})_{\mathbb{Q}} \simeq \text{Coh}(\mathcal{O}_{\mathcal{S}_K})$ is locally free of finite rank, and Ψ is an isomorphism $F^*\mathcal{E} \xrightarrow{\cong} \mathcal{E}$.

An object (\mathcal{E}, Ψ) in $F\text{-Vect}([\overline{\mathcal{Y}}^{\text{sm}}/G_Y]_K)$ (resp. $F\text{-Vect}((\overline{\mathcal{X}}, \mathcal{Z})_K^{1/n})$) is called generically semistable (gss) if, for any dense open formal subscheme $\mathcal{U}_o \subseteq \mathcal{X}_o$, the image $(\mathcal{E}|_{\mathcal{U}_K}, \Psi|_{\mathcal{U}_K})$ of (\mathcal{E}, Ψ) by the restriction functor

$$\begin{aligned} F\text{-Vect}([\overline{\mathcal{Y}}^{\text{sm}}/G_Y]_K) &\longrightarrow F\text{-Vect}((\mathcal{U} \times_{\mathcal{X}} [\overline{\mathcal{Y}}^{\text{sm}}/G_Y])_K) \xrightarrow{\cong} F\text{-Vect}(\mathcal{U}_K)_{\mathbb{Q}} \\ (\text{resp. } F\text{-Vect}((\overline{\mathcal{X}}, \mathcal{Z})_K^{1/n}) &\longrightarrow F\text{-Vect}((\mathcal{U} \times_{\mathcal{X}} (\overline{\mathcal{X}}, \mathcal{Z})_K^{1/n})_K) \xrightarrow{\cong} F\text{-Vect}(\mathcal{U}_K)) \end{aligned}$$

admits no non-trivial saturated subobject (\mathcal{E}', Ψ') with $\mu(\mathcal{E}') < \mu(\mathcal{E}|_{\mathcal{U}_K})$. (Note that $\mu(\mathcal{E}|_{\mathcal{U}_K})$ does not depend on \mathcal{U} . Hence we denote it by $\mu(\mathcal{E})$.) We denote the full subcategory of $F\text{-Vect}([\overline{\mathcal{Y}}^{\text{sm}}/G_Y]_K)$ (resp. $F\text{-Vect}((\overline{\mathcal{X}}, \mathcal{Z})_K^{1/n})$) consisting of generically semistable objects (\mathcal{E}, Ψ) with $\mu(\mathcal{E}) = 0$ by $F\text{-Vect}([\overline{\mathcal{Y}}^{\text{sm}}/G_Y]_K)^{\text{gss}, \mu=0}$ (resp. $F\text{-Vect}((\overline{\mathcal{X}}, \mathcal{Z})_K^{1/n})^{\text{gss}, \mu=0}$). Then we have the following proposition:

Proposition 5.22. *Let the notations be as above and assume that $\dim X = 1$. Then we have the canonical equivalences*

$$(5.12) \quad \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} F\text{-Latt}([\overline{\mathcal{Y}}^{\text{sm}}/G_Y]_{\mathbb{Q}})^{\circ} \xrightarrow{\cong} \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} F\text{-Vect}([\overline{\mathcal{Y}}^{\text{sm}}/G_Y]_K)^{\text{gss}, \mu=0},$$

$$(5.13) \quad \varinjlim_{(n,p)=1} F\text{-Latt}((\overline{\mathcal{X}}, \mathcal{Z})_K^{1/n})^{\circ} \xrightarrow{\cong} \varinjlim_{(n,p)=1} F\text{-Vect}((\overline{\mathcal{X}}, \mathcal{Z})_K^{1/n})^{\text{gss}, \mu=0}.$$

Proof. Let $\overline{\mathcal{Y}}_{\bullet}^{\text{sm}}$ be as in the proof of Theorem 4.1, and in the proof, we omit to write the superscript sm . (This is justified because we have $\overline{\mathcal{Y}} = \overline{\mathcal{Y}}^{\text{sm}}$ in the case of curves.) Then we have the diagram

$$\begin{aligned} (5.14) \quad \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} F\text{-Vect}([\overline{\mathcal{Y}}/G_Y]_K)^{\text{gss}, \mu=0} &\longrightarrow \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} F\text{-Vect}(\overline{\mathcal{Y}}_{\bullet})^{\text{gss}, \mu=0} \\ &\longleftarrow \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} \{F\text{-Latt}(\overline{\mathcal{Y}}_m)_{\mathbb{Q}}^{\circ}\}_{m=0,1,2} \\ &\longleftarrow \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} F\text{-Latt}(\overline{\mathcal{Y}}_{\bullet})_{\mathbb{Q}}^{\circ} \\ &\xleftarrow{\cong} F\text{-Latt}([\overline{\mathcal{Y}}/G_Y]_{\mathbb{Q}})^{\circ}, \end{aligned}$$

where $F\text{-Vect}(\overline{\mathcal{Y}}_{\bullet})^{\text{gss}, \mu=0}$ denotes the full subcategory of $F\text{-Vect}(\overline{\mathcal{Y}}_{\bullet})$ consisting of objects whose restriction to $F\text{-Vect}(\overline{\mathcal{Y}}_m)$ is contained in $F\text{-Vect}(\overline{\mathcal{Y}}_m)^{\text{gss}, \mu=0}$ for each m .

In the diagram (5.14), the third arrow is an equivalence since we already proved it in the proof of Proposition 5.18, and the second arrow is an equivalence by Proposition 5.21. So, to show the equivalence (5.12), it suffices to prove that the first arrow in (5.14) is an equivalence. Since it is easy to see that the condition ‘gss and $\mu = 0$ ’ is preserved by the natural functor

$$(5.15) \quad F\text{-Vect}([\overline{\mathcal{Y}}/G_Y]_K) \longrightarrow F\text{-Vect}(\overline{\mathcal{Y}}_\bullet),$$

it suffices to show that the functor (5.15) is an equivalence. Let $\{\text{Coh}(\overline{\mathcal{Y}}_m)_\mathbb{Q}\}_{m=0,1,2}$ be the category of compatible family of objects in $\text{Coh}(\overline{\mathcal{Y}}_m)_\mathbb{Q}$ ($m = 0, 1, 2$). Then, the functor (5.15) is induced by the functor

$$(5.16) \quad \text{Coh}([\overline{\mathcal{Y}}/G_Y])_\mathbb{Q} \longrightarrow \{\text{Coh}(\overline{\mathcal{Y}}_m)_\mathbb{Q}\}_{m=0,1,2},$$

as the locally free part of (5.16) with F -structure. Since local freeness for an object in $\text{Coh}([\overline{\mathcal{Y}}/G_Y])_\mathbb{Q}$ can be checked in $\text{Coh}(\overline{\mathcal{Y}}_m)_\mathbb{Q}$ ($m = 0, 1, 2$), it suffices to show the equivalence of the functor (5.16). Note that it is factorized as

$$\text{Coh}([\overline{\mathcal{Y}}/G_Y])_\mathbb{Q} \longrightarrow \text{Coh}(\overline{\mathcal{Y}}_\bullet)_\mathbb{Q} \longrightarrow \{\text{Coh}(\overline{\mathcal{Y}}_m)_\mathbb{Q}\}_{m=0,1,2},$$

in which the first arrow is an equivalence by usual faithfully flat descent. Moreover, we see by the same way as [35, 1.9] that the second arrow is also an equivalence. Hence we have shown the equivalence (5.12). We can prove the equivalence (5.13) in the same way, by using $\overline{\mathcal{X}}_\bullet^{(n)}$ in the proof of Theorem 4.1 instead of $\overline{\mathcal{Y}}_\bullet$. \square

Next, let the notations be as in Definition 4.3. Then ‘a parabolic vector bundle on $(\overline{\mathcal{X}}, \mathcal{Z})_K$ ’ is defined to be an inductive system $(\mathcal{E}_\alpha)_{\alpha \in \mathbb{Z}_{(p)}}$ of locally free $\mathcal{O}_{\overline{\mathcal{X}}_K}$ -modules of finite rank satisfying the following conditions:

- (a) For any $1 \leq i \leq r$, there is an isomorphism as inductive systems

$$((\mathcal{E}_{\alpha+e_i})_\alpha, (\iota_{\alpha+e_i, \beta+e_i})_{\alpha, \beta}) \cong ((\mathcal{E}_\alpha(\mathcal{Z}_{i,K}))_\alpha, (\iota_{\alpha\beta} \otimes \text{id})_{\alpha, \beta})$$

via which the morphism $(\iota_{\alpha, \alpha+e_i})_\alpha : (\mathcal{E}_\alpha)_\alpha \longrightarrow (\mathcal{E}_{\alpha+e_i})_\alpha$ is identified with the morphism $\text{id} \otimes \iota_{0, e_i}^0 : (\mathcal{E}_\alpha)_\alpha \longrightarrow (\mathcal{E}_\alpha(\mathcal{Z}_{i,K}))_\alpha$, where $\iota_{0, e_i}^0 : \mathcal{O}_{\overline{\mathcal{X}}} \hookrightarrow \mathcal{O}_{\overline{\mathcal{X}}}(\mathcal{Z}_{i,K})$ denotes the natural inclusion.

- (b) There exists a positive integer n prime to p which satisfies the following condition: For any $\alpha = (\alpha_i)_i$, $\iota_{\alpha'\alpha}$ is an isomorphism if we put $\alpha' = ([n\alpha_i]/n)_i$.

A parabolic F -vector bundle on $(\overline{\mathcal{X}}, \mathcal{Z})_K$ is a pair $((\mathcal{E}_\alpha)_\alpha, (\Psi_\alpha)_\alpha)$ consisting of a parabolic vector bundle $(\mathcal{E}_\alpha)_\alpha$ on $(\overline{\mathcal{X}}, \mathcal{Z})_K$ endowed with morphisms $\Psi_\alpha : F^*\mathcal{E}_\alpha \longrightarrow \mathcal{E}_{q\alpha}$ in the category of $\mathcal{O}_{\overline{\mathcal{X}}_K}$ -modules such that $\varinjlim_\alpha \Psi_\alpha : \varinjlim_\alpha F^*\mathcal{E}_\alpha \longrightarrow \varinjlim_\alpha \mathcal{E}_{q\alpha}$ is isomorphic as ind-objects.

For $\alpha := (\alpha_i)_i \in \mathbb{Z}_{(p)}^r$, let $\mathcal{O}_{\overline{\mathcal{X}}_K}(\sum_i \alpha_i \mathcal{Z}_{i,K}) := (\mathcal{O}_{\overline{\mathcal{X}}}(\sum_i \alpha_i \mathcal{Z}_i)_\beta)_\beta$ be the parabolic vector bundle on $(\overline{\mathcal{X}}, \mathcal{Z})_K$ defined by $\mathcal{O}_{\overline{\mathcal{X}}}(\sum_i \alpha_i \mathcal{Z}_{i,K})_\beta := \mathcal{O}_{\overline{\mathcal{X}}}(\sum_i [\alpha_i + \beta_i] \mathcal{Z}_{i,K})$ (where

$\beta = (\beta_i)_i$, and we say that a parabolic F -vector bundle $((\mathcal{E}_\alpha)_\alpha, (\Psi_\alpha)_\alpha)$ on $(\overline{\mathcal{X}}, \mathcal{Z})_K$ is locally abelian if there exists some positive integer n prime to p and an admissible covering $\{\mathfrak{X}_\lambda\}_\lambda$ of $\mathcal{X}_{n,K} := (\mathcal{X} \otimes_{O_K} O_K[\mu_n])_K$ such that $(\mathcal{E}_\alpha)_\alpha|_{\mathfrak{X}_\lambda}$ has the form $\bigoplus_{j=1}^\mu \mathcal{O}_{\overline{\mathcal{X}}_K}(\sum_i \alpha_{ij} \mathcal{Z}_{i,K})|_{\mathfrak{X}_\lambda}$ for some $\alpha_{ij} \in \mathbb{Z}_{(p)}$, for each λ . We denote the category of locally abelian parabolic F -vector bundles on $(\overline{\mathcal{X}}, \mathcal{Z})_K$ by $\text{Par-}F\text{-Vect}((\overline{\mathcal{X}}, \mathcal{Z})_K)$.

An object $((\mathcal{E}_\alpha)_\alpha, (\Psi_\alpha)_\alpha)$ in $\text{Par-}F\text{-Vect}((\overline{\mathcal{X}}, \mathcal{Z})_K)$ is called generically semistable (gss) if, for any dense open formal subscheme $\mathcal{U}_\circ \subseteq \mathcal{X}_\circ$, the image $(\mathcal{E}|_{\mathcal{U}_K}, \Psi|_{\mathcal{U}_K})$ of $((\mathcal{E}_\alpha)_\alpha, (\Psi_\alpha)_\alpha)$ by the restriction functor

$$\text{Par-}F\text{-Vect}((\overline{\mathcal{X}}, \mathcal{Z})_K) \longrightarrow \text{Par-}F\text{-Vect}((\mathcal{U} \times_{\overline{\mathcal{X}}} (\overline{\mathcal{X}}, \mathcal{Z}))_K) \xrightarrow{=} F\text{-Vect}(\mathcal{U}_K)$$

(where $\mathcal{U}_K := (\mathcal{U}_\circ \otimes_{W(k)} O_K)_K$) admits no non-trivial saturated subobject (\mathcal{E}', Ψ') with $\mu(\mathcal{E}') < \mu(\mathcal{E}|_{\mathcal{U}_K})$. (Note that $\mu(\mathcal{E}|_{\mathcal{U}_K})$ does not depend on \mathcal{E} . Hence we denote it by $\mu(\mathcal{E})$.) We denote the full subcategory of $\text{Par-}F\text{-Vect}((\overline{\mathcal{X}}, \mathcal{Z})_K)$ consisting of generically semistable objects (\mathcal{E}, Ψ) with $\mu(\mathcal{E}) = 0$ by $\text{Par-}F\text{-Vect}((\overline{\mathcal{X}}, \mathcal{Z})_K)^{\text{gss}, \mu=0}$. Then we have the following proposition.

Proposition 5.23. *Let the notations be as above and assume that $\dim X = 1$. Then we have the canonical equivalence of categories*

$$(5.17) \quad \text{Par-}F\text{-Latt}(\overline{\mathcal{X}}, \mathcal{Z})_K^\circ \xrightarrow{=} \text{Par-}F\text{-Vect}((\overline{\mathcal{X}}, \mathcal{Z})_K)^{\text{gss}, \mu=0}.$$

Proof. We can prove the equivalence

$$(5.18) \quad \mathbf{a} : \varinjlim_{(n,p)=1} F\text{-Vect}((\overline{\mathcal{X}}, \mathcal{Z})_K^{1/n}) \xrightarrow{=} \text{Par-}F\text{-Vect}((\overline{\mathcal{X}}, \mathcal{Z})_K)$$

in the same way as Theorem 4.5 and the generic semistabilities and the values of μ coincide via the above equivalence. (In the proof of Theorem 4.5, when we are given an object \mathcal{E} in $\text{Vect}((\overline{\mathcal{X}}, \mathcal{Z})_K^{1/n})$, an open affine $\mathcal{U} \subseteq \mathcal{X}$ and a closed point x of \mathcal{U} , we constructed an open formal subscheme $\mathcal{U}_x = \{f_x \neq 0\}$ of \mathcal{U} containing x on which $\mathbf{a}(\mathcal{E})$ has a simple shape. Here, for an object \mathcal{E} in $\text{Vect}((\overline{\mathcal{X}}, \mathcal{Z})_K^{1/n})$, an open affine $\mathcal{U} \subseteq \mathcal{X}$ and a point x of \mathcal{U}_K , we can construct in the same way an open rigid subspace $\mathfrak{U}_x = \{f_x \neq 0\}$ of \mathcal{U}_K containing x on which $\mathbf{a}(\mathcal{E})$ has a simple shape, and we see that the covering $\mathcal{X}_K = \bigcup_{\mathcal{U}:\text{affine}} \bigcup_{x \in \mathcal{U}_K} \mathfrak{U}_x$ is an admissible covering.) Then the desired equivalence follows from this, (5.13) and Theorem 4.5. \square

Remark 5.24. (1) As we see from the above proof, the equivalence (5.18) holds for X of any dimension.

(2) When $\dim X = 1$, any parabolic F -vector bundle on $(\overline{\mathcal{X}}, \mathcal{Z})_K$ is locally abelian. Hence we can drop the condition of locally abelianness from the definition of $\text{Par-}F\text{-Vect}((\overline{\mathcal{X}}, \mathcal{Z})_K)$ when $\dim X = 1$.

As an immediate consequence of Propositions 5.21, 5.22 and 5.23, we obtain the following, which is a p -adic analogues (F -vector bundle version) of (0.2) and (0.3) and (0.5) which includes the notion of stability and in which neither isocrystal structure and lattice structure appear, in the case of curves with strong liftability condition:

Corollary 5.25. *Let the notations be as above and assume that $\dim X = 1$. Then we have the equivalences*

$$\begin{aligned} \mathrm{Rep}_{K^\sigma}(\pi_1^t(X)) &\xrightarrow{=} \varinjlim_{Y \rightarrow X \in \mathcal{G}_X^t} F\text{-Vect}([\overline{\mathcal{Y}}/G_Y]_K)^{\mathrm{gss}, \mu=0}. \\ \mathrm{Rep}_{K^\sigma}(\pi_1^t(X)) &\xrightarrow{=} \varinjlim_{(n,p)=1} F\text{-Vect}((\overline{\mathcal{X}}, \mathcal{Z})_K^{1/n})^{\mathrm{gss}, \mu=0}. \\ \mathrm{Rep}_{K^\sigma}(\pi_1^t(X)) &\xrightarrow{=} \mathrm{Par}\text{-}F\text{-Vect}((\overline{\mathcal{X}}, \mathcal{Z})_K)^{\mathrm{gss}, \mu=0}. \end{aligned}$$

This is a precise form of the micro reciprocity law conjectured in [51, 48.6, 49.3].

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